

# FOURIER MULTIPLIERS, SYMBOLS AND NUCLEARITY ON COMPACT MANIFOLDS

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**ABSTRACT.** The notion of invariant operators, or Fourier multipliers, is discussed for densely defined operators on Hilbert spaces, with respect to a fixed partition of the space into a direct sum of finite dimensional subspaces. As a consequence, given a compact manifold  $M$  endowed with a positive measure, we introduce a notion of the operator's full symbol adapted to the Fourier analysis relative to a fixed elliptic operator  $E$ . We give a description of Fourier multipliers, or of operators invariant relative to  $E$ . We apply these concepts to study Schatten classes of operators on  $L^2(M)$  and to obtain a formula for the trace of trace class operators. We also apply it to provide conditions for operators between  $L^p$ -spaces to be  $r$ -nuclear in the sense of Grothendieck.

## 1. INTRODUCTION

Let  $M$  be a closed manifold (i.e. a compact smooth manifold without boundary) of dimension  $n$  endowed with a positive measure  $dx$ . Given an elliptic positive pseudo-differential operator  $E$  of order  $\nu$  on  $M$ , by considering an orthonormal basis consisting of eigenfunctions of  $E$  we will associate a discrete Fourier analysis to the operator  $E$  in the sense introduced by Seeley ([See65], [See69]). This analysis allows us to introduce further a notion of invariant operators and of matrix-symbols corresponding to those operators. The operators on  $M$  will be then analysed in terms of the corresponding symbols relative to the operator  $E$ .

As a general framework, we first discuss invariant operators, or Fourier multipliers in a general Hilbert space  $\mathcal{H}$ . This notion is based on a partition of  $\mathcal{H}$  into a direct sum of finite dimensional subspaces, so that a densely defined operator on  $\mathcal{H}$  can be decomposed as acting in these subspaces. There are two main examples of this construction discussed in the paper: operators on  $\mathcal{H} = L^2(M)$  for a compact manifold  $M$  as well as operators on  $\mathcal{H} = L^2(G)$  for a compact Lie group  $G$ . The difference in approaches to these settings is in the choice of partitions of  $\mathcal{H}$  into direct sums of subspaces: in the former case they are chosen as eigenspaces of a fixed elliptic pseudo-differential operator on  $M$  while in the latter case they are chosen as linear spans of matrix coefficients of inequivalent irreducible unitary representations of  $G$ .

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We note that for some results, the self-adjointness and ellipticity of  $E$  can be dropped, see [RT15].

We give applications of these notions to the derivation of conditions characterising those invariant operators on  $L^2(M)$  that belong to Schatten classes. Furthermore, we also give conditions for nuclearity on  $L^p$ -spaces and, more generally, for the  $r$ -nuclearity of operators. While the theory of  $r$ -nuclear operators in general Banach spaces has been developed by Grothendieck [Gro55] with numerous further advances (e.g. in [HP10, Kön78, Olo72, Pie84, RL13]), in this paper we give conditions in terms of symbols for operators to be  $r$ -nuclear from  $L^{p_1}(M)$  to  $L^{p_2}(M)$  for  $1 \leq p_1, p_2 < \infty$  and  $0 < r \leq 1$ . Consequently, we determine relations between  $p_1, p_2, r$  and  $\alpha$  ensuring that the powers  $(I + E)^{-\alpha}$  are  $r$ -nuclear. Trace formulas are also obtained relating operator traces to expressions involving their symbols.

In the recent work [DR14c] the authors found sufficient conditions for operators to belong to Schatten classes  $S_p$  on compact manifolds in terms of their Schwartz integral kernels. For  $p < 2$ , it is customary to impose regularity conditions on the kernel because there are counterexamples to conditions formulated only in terms of the integrability of kernels. Such examples go back to Carleman's work [Car16] and their relevance to Schatten classes has been discussed in [DR14b]. A characteristic feature of conditions of this paper is that no regularity is assumed neither on the symbol nor on the kernel. In the case of compact Lie groups, our results extend results on Schatten classes and on  $r$ -nuclear operators on  $L^p$  spaces that have been obtained in [DR13] and [DR14b]. We show this by relating the symbols introduced in this paper to matrix-valued symbols on compact Lie groups developed in [RT13] and in [RT10].

Schatten classes of pseudo-differential operators in the setting of the Weyl-Hörmander calculus have been considered in [Tof06], [Tof08], [BN04], [BN07], [BT10]. Conditions for symbols of lower regularity we given in [Sob14]. For the global analysis of pseudo-differential operators on  $\mathbb{R}^n$  see [BBR96], as well as [NR10, Chapter 4] also for the basic general introduction to Schatten classes.

To formulate the notions more precisely, let  $\mathcal{H}$  be a complex Hilbert space and let  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a linear compact operator. If we denote by  $T^* : \mathcal{H} \rightarrow \mathcal{H}$  the adjoint of  $T$ , then the linear operator  $(T^*T)^{\frac{1}{2}} : \mathcal{H} \rightarrow \mathcal{H}$  is positive and compact. Let  $(\psi_k)_k$  be an orthonormal basis for  $\mathcal{H}$  consisting of eigenvectors of  $|T| = (T^*T)^{\frac{1}{2}}$ , and let  $s_k(T)$  be the eigenvalue corresponding to the eigenvector  $\psi_k$ ,  $k = 1, 2, \dots$ . The non-negative numbers  $s_k(T)$ ,  $k = 1, 2, \dots$ , are called the singular values of  $T : \mathcal{H} \rightarrow \mathcal{H}$ . If  $0 < p < \infty$  and the sequence of singular values is  $p$ -summable, then  $T$  is said to belong to the Schatten class  $S_p(\mathcal{H})$ , and it is well known that each  $S_p(\mathcal{H})$  is an ideal in  $\mathcal{L}(\mathcal{H})$ . If  $1 \leq p < \infty$ , a norm is associated to  $S_p(\mathcal{H})$  by

$$\|T\|_{S_p} = \left( \sum_{k=1}^{\infty} (s_k(T))^p \right)^{\frac{1}{p}}.$$

If  $1 \leq p < \infty$  the class  $S_p(\mathcal{H})$  becomes a Banach space endowed by the norm  $\|T\|_{S_p}$ . If  $p = \infty$  we define  $S_{\infty}(\mathcal{H})$  as the class of bounded linear operators on  $\mathcal{H}$ , with  $\|T\|_{S_{\infty}} := \|T\|_{op}$ , the operator norm. For the Schatten class  $S_2$  we will sometimes write  $\|T\|_{\text{HS}}$  instead of  $\|T\|_{S_2}$ . In the case  $0 < p < 1$  the quantity  $\|T\|_{S_p}$  only defines

a quasi-norm, and  $S_p(\mathcal{H})$  is also complete. The space  $S_1(\mathcal{H})$  is known as the *trace class* and an element of  $S_2(\mathcal{H})$  is usually said to be a *Hilbert-Schmidt* operator. For the basic theory of Schatten classes we refer the reader to [GK69], [RS75], [Sim79], [Sch70].

It is well known that the class  $S_2(L^2)$  is characterised by the square integrability of the corresponding integral kernels, however, kernel estimates of this type are not effective for classes  $S_p(L^2)$  with  $p < 2$ . This is explained by a classical Carleman's example [Car16] on the summability of Fourier coefficients of continuous functions (see [DR14b] for a complete explanation of this fact). This obstruction explains the relevance of symbolic Schatten criteria and here we will clarify the advantage of the symbol approach with respect to this obstruction. With this approach, no regularity of the kernel needs to be assumed.

In Section 6 we discuss the relation of our approach to that of the global analysis on compact Lie groups. In particular, in the case of compact Lie groups the Fourier coefficients can be arranged into a (square) matrix rather than in a column leading to several simplifications. On general compact manifolds, this is not possible since the multiplicities  $d_j$  do not need to be all squares of integers.

We introduce  $\ell^p$ -style norms on the space of symbols  $\Sigma$ , yielding discrete spaces  $\ell^p(\Sigma)$  for  $0 < p \leq \infty$ , normed for  $p \geq 1$ . Denoting by  $\sigma_T$  the matrix symbol of an invariant operator  $T$  provided by Theorem 4.1, Schatten classes of invariant operators on  $L^2(M)$  can be characterised concisely by conditions

$$T \in \mathcal{L}(L^2(M)) \iff \sigma_T \in \ell^\infty(\Sigma), \quad (1.1)$$

and for  $0 < p < \infty$ ,

$$T \in S_p(L^2(M)) \iff \sigma_T \in \ell^p(\Sigma), \quad (1.2)$$

see (7.4) and (7.5). Here, the condition that  $T$  is invariant will mean that  $T$  is strongly commuting with  $E$  (see Theorem 4.1). On the level of the Fourier transform this means that

$$\widehat{Tf}(\ell) = \sigma(\ell)\widehat{f}(\ell)$$

for a family of matrices  $\sigma(\ell)$ , i.e.  $T$  assumes the familiar form of a Fourier multiplier.

In Section 2 in Theorem 2.1 we discuss the abstract notion of symbol for operators densely defined in a general Hilbert space  $\mathcal{H}$ , and give several alternative formulations for invariant operators, or for Fourier multipliers, relative to a fixed partition of  $\mathcal{H}$  into a direct sum of finite dimensional subspaces,

$$\mathcal{H} = \bigoplus_j H_j.$$

Consequently, in Theorem 2.3 we give the necessary and sufficient condition for the bounded extendability of an invariant operator to  $\mathcal{L}(\mathcal{H})$  in terms of its symbol, and in Theorem 2.5 the necessary and sufficient condition for the operator to be in Schatten classes  $S_r(\mathcal{H})$  for  $0 < r < \infty$ , as well as the trace formula for operators in the trace class  $S_1(\mathcal{H})$  in terms of their symbols. As our subsequent analysis relies to a large extent on properties of elliptic pseudo-differential operators on  $M$ , in Sections 3 and 4 we specify this abstract analysis to the setting of operators densely defined on  $L^2(M)$ . The main difference is that we now adopt the Fourier analysis to a

fixed elliptic positive pseudo-differential operator  $E$  on  $M$ , contrary to the case of an operator  $E_o \in \mathcal{L}(\mathcal{H})$  in Theorem 2.2.

The notion of invariance depends on the choice of the spaces  $H_j$ . Thus, in the analysis of operators on  $M$  we take  $H_j$ 's to be the eigenspaces of  $E$ . However, other choices are possible. For example, for  $\mathcal{H} = L^2(G)$  for a compact Lie group  $G$ , choosing  $H_j$ 's as linear spans of representation coefficients for inequivalent irreducible unitary representations of  $G$ , we make a link to the quantization of pseudo-differential operator on compact Lie groups as in [RT10]. These two partitions coincide when inequivalent representations of  $G$  produce distinct eigenvalues of the Laplacian; for example, this is the case for  $G = \mathrm{SO}(3)$ . However, the partitions are different when inequivalent representations produce equal eigenvalues, which is the case, for example, for  $G = \mathrm{SO}(4)$ . For the more explicit example on  $\mathcal{H} = L^2(\mathbb{T}^n)$  on the torus see Remark 2.6. A similar choice could be made in other settings producing a discrete spectrum and finite dimensional eigenspaces, for example for operators in Shubin classes on  $\mathbb{R}^n$ , see Chodosh [Cho11] for the case  $n = 1$ .

The analogous concept to Schatten classes in the setting of Banach spaces is the notion of  $r$ -nuclearity introduced by Grothendieck [Gro55]. It has applications to questions of the distribution of eigenvalues of operators in Banach spaces. In the setting of compact Lie groups these applications have been discussed in [DR14b] and they include conclusions on the distribution or summability of eigenvalues of operators acting on  $L^p$ -spaces. Another application is the Grothendieck-Lidskii formula which is the formula for the trace of operators on  $L^p(M)$ . Once we have  $r$ -nuclearity, most of further arguments are then purely functional analytic, so they apply equally well in the present setting of closed manifolds. Because of this we omit the repetition of statements and refer the reader to [DR14b] for further such applications.

Some results of this paper have been announced in [DR14a], so here we provide their proofs, including a correction to the formulation of [DR14a, Theorem 3.1, (iv)] given by Theorem 4.1, (iv), of this paper.

The paper is organised as follows. In Section 2 we discuss Fourier multipliers and their symbols in general Hilbert spaces. In Section 3 we associate a global Fourier analysis to an elliptic positive pseudo-differential operator  $E$  on a closed manifold  $M$ . In Section 4 we introduce the class of operators invariant relative to  $E$  as well as their matrix-valued symbols, and apply this to characterise invariant operators in Schatten classes in Section 5. In Section 6 we relate the analysis developed so far to the analysis on compact Lie groups from [RT13], [RT10], and establish formula relating their matrix symbols in the case when  $M$  is a compact Lie group. In particular, we will see that left-invariant operators on compact Lie groups are invariant in our sense. In Section 7 we analyse the integral kernels of invariant operators on general closed manifolds. Finally, in Section 8 we apply our analysis to study  $r$ -nuclear operators on  $L^p$ -spaces.

Throughout the paper, we denote  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . Also  $\delta_{j\ell}$  will denote the Kronecker delta, i.e.  $\delta_{j\ell} = 1$  for  $j = \ell$ , and  $\delta_{j\ell} = 0$  for  $j \neq \ell$ .

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## 2. FOURIER MULTIPLIERS IN HILBERT SPACES

In this section we present an abstract set up to describe what we will call invariant operators, or Fourier multipliers, acting on a general Hilbert space  $\mathcal{H}$ . We will give several characterisations of such operators and their symbols. Consequently, we will apply these notions to describe several properties of the operators, in particular, their boundedness on  $\mathcal{H}$  as well as the Schatten properties.

We note that direct integrals (sums in our case) of Hilbert spaces have been investigated in a much greater generality, see e.g. Bruhat [Bru68], Dixmier [Dix96, Ch 2., §2], [Dix77, Appendix]. The setting required for our analysis is much simpler, so we prefer to adapt it specifically for consequent applications, also providing short proofs for our statements.

The main application of the constructions below will be in the setting when  $M$  is a compact manifold without boundary,  $\mathcal{H} = L^2(M)$  and  $\mathcal{H}^\infty = C^\infty(M)$ , which will be described in detail in Section 3. However, several facts can be more clearly interpreted in the setting of abstract Hilbert spaces, which will be our set up in this section. With this particular example in mind, in the following theorem, we can think of  $\{e_j^k\}$  being an orthonormal basis given by eigenfunctions of an elliptic operator on  $M$ , and  $d_j$  the corresponding multiplicities. However, we allow flexibility in grouping the eigenfunctions in order to be able to also cover the case of operators on compact Lie groups.

**Theorem 2.1.** *Let  $\mathcal{H}$  be a complex Hilbert space and let  $\mathcal{H}^\infty \subset \mathcal{H}$  be a dense linear subspace of  $\mathcal{H}$ . Let  $\{d_j\}_{j \in \mathbb{N}_0} \subset \mathbb{N}$  and let  $\{e_j^k\}_{j \in \mathbb{N}_0, 1 \leq k \leq d_j}$  be an orthonormal basis of  $\mathcal{H}$  such that  $e_j^k \in \mathcal{H}^\infty$  for all  $j$  and  $k$ . Let  $H_j := \text{span}\{e_j^k\}_{k=1}^{d_j}$ , and let  $P_j : \mathcal{H} \rightarrow H_j$  be the orthogonal projection. For  $f \in \mathcal{H}$ , we denote*

$$\widehat{f}(j, k) := (f, e_j^k)_{\mathcal{H}}$$

*and let  $\widehat{f}(j) \in \mathbb{C}^{d_j}$  denote the column of  $\widehat{f}(j, k)$ ,  $1 \leq k \leq d_j$ . Let  $T : \mathcal{H}^\infty \rightarrow \mathcal{H}$  be a linear operator. Then the following conditions are equivalent:*

- (A) *For each  $j \in \mathbb{N}_0$ , we have  $T(H_j) \subset H_j$ .*
- (B) *For each  $\ell \in \mathbb{N}_0$  there exists a matrix  $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$  such that for all  $e_j^k$*

$$\widehat{Te_j^k}(\ell, m) = \sigma(\ell)_{mk} \delta_{j\ell}.$$

- (C) *If in addition,  $e_j^k$  are in the domain of  $T^*$  for all  $j$  and  $k$ , then for each  $\ell \in \mathbb{N}_0$  there exists a matrix  $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$  such that*

$$\widehat{Tf}(\ell) = \sigma(\ell) \widehat{f}(\ell)$$

*for all  $f \in \mathcal{H}^\infty$ .*

*The matrices  $\sigma(\ell)$  in (B) and (C) coincide.*

*The equivalent properties (A)–(C) follow from the condition*

- (D) *For each  $j \in \mathbb{N}_0$ , we have  $TP_j = P_jT$  on  $\mathcal{H}^\infty$ .*

*If, in addition,  $T$  extends to a bounded operator  $T \in \mathcal{L}(\mathcal{H})$  then (D) is equivalent to (A)–(C).*

Under the assumptions of Theorem 2.1, we have the direct sum decomposition

$$\mathcal{H} = \bigoplus_{j=0}^{\infty} H_j, \quad H_j = \text{span}\{e_j^k\}_{k=1}^{d_j}, \quad (2.1)$$

and we have  $d_j = \dim H_j$ . The two applications that we will consider will be with  $\mathcal{H} = L^2(M)$  for a compact manifold  $M$  with  $H_j$  being the eigenspaces of an elliptic pseudo-differential operator  $E$ , or with  $\mathcal{H} = L^2(G)$  for a compact Lie group  $G$  with

$$H_j = \text{span}\{\xi_{km}\}_{1 \leq k, m \leq d_\xi}$$

for a unitary irreducible representation  $\xi \in [\xi_j] \in \widehat{G}$ . The difference is that in the first case we will have that the eigenvalues of  $E$  corresponding to  $H_j$ 's are all distinct, while in the second case the eigenvalues of the Laplacian on  $G$  for which  $H_j$ 's are the eigenspaces, may coincide. In Remark 2.6 we give an example of this difference for operators on the torus  $\mathbb{T}^n$ .

In view of properties (A) and (C), respectively, an operator  $T$  satisfying any of the equivalent properties (A)–(C) in Theorem 2.1, will be called an *invariant operator*, or a *Fourier multiplier relative to the decomposition*  $\{H_j\}_{j \in \mathbb{N}_0}$  in (2.1). If the collection  $\{H_j\}_{j \in \mathbb{N}_0}$  is fixed once and for all, we can just say that  $T$  is *invariant* or a *Fourier multiplier*.

The family of matrices  $\sigma$  will be called the *matrix symbol of  $T$  relative to the partition  $\{H_j\}$  and to the basis  $\{e_j^k\}$* . It is an element of the space  $\Sigma$  defined by

$$\Sigma = \{\sigma : \mathbb{N}_0 \ni \ell \mapsto \sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}\}. \quad (2.2)$$

A criterion for the extendability of  $T$  to  $\mathcal{L}(\mathcal{H})$  in terms of its symbol will be given in Theorem 2.3.

For  $f \in \mathcal{H}$ , in the notation of Theorem 2.1, by definition we have

$$f = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \widehat{f}(j, k) e_j^k \quad (2.3)$$

with the convergence of the series in  $\mathcal{H}$ . Since  $\{e_j^k\}_{j \geq 0}^{1 \leq k \leq d_j}$  is a complete orthonormal system on  $\mathcal{H}$ , for all  $f \in \mathcal{H}$  we have the Plancherel formula

$$\|f\|_{\mathcal{H}}^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |(f, e_j^k)|^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |\widehat{f}(j, k)|^2 = \|\widehat{f}\|_{\ell^2(\mathbb{N}_0, \Sigma)}^2, \quad (2.4)$$

where we interpret  $\widehat{f} \in \Sigma$  as an element of the space

$$\ell^2(\mathbb{N}_0, \Sigma) = \{h : \mathbb{N}_0 \rightarrow \prod_d \mathbb{C}^d : h(j) \in \mathbb{C}^{d_j} \text{ and } \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |h(j, k)|^2 < \infty\}, \quad (2.5)$$

and where we have written  $h(j, k) = h(j)_k$ . In other words,  $\ell^2(\mathbb{N}_0, \Sigma)$  is the space of all  $h \in \Sigma$  such that

$$\sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |h(j, k)|^2 < \infty.$$

We endow  $\ell^2(\mathbb{N}_0, \Sigma)$  with the norm

$$\|h\|_{\ell^2(\mathbb{N}_0, \Sigma)} := \left( \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |h(j, k)|^2 \right)^{\frac{1}{2}}. \quad (2.6)$$

We note that the matrix symbol  $\sigma(\ell)$  depends not only on the partition (2.1) but also on the choice of the orthonormal basis. Whenever necessary, we will indicate the dependance of  $\sigma$  on the orthonormal basis by writing  $(\sigma, \{e_j^k\}_{j \geq 0}^{1 \leq k \leq d_j})$  and we also will refer to  $(\sigma, \{e_j^k\}_{j \geq 0}^{1 \leq k \leq d_j})$  as the *symbol* of  $T$ . Throughout this section the orthonormal basis will be fixed and unless there is some risk of confusion the symbols will be denoted simply by  $\sigma$ . In the invariant language, as will be clear from the proof of Theorem 2.1, we have that the transpose of the symbol,  $\sigma(j)^\top = T|_{H_j}$  is just the restriction of  $T$  to  $H_j$ , which is well defined in view of the property (A).

We will also sometimes write  $T_\sigma$  to indicate that  $T_\sigma$  is an operator corresponding to the symbol  $\sigma$ . It is clear from the definition that invariant operators are uniquely determined by their symbols. Indeed, if  $T = 0$  we obtain  $\sigma = 0$  for any choice of an orthonormal basis. Moreover, we note that by taking  $j = \ell$  in (B) of Theorem 2.1 we obtain the formula for the symbol:

$$\sigma(j)_{mk} = \widehat{Te_j^k}(j, m), \quad (2.7)$$

for all  $1 \leq k, m \leq d_j$ . The formula (2.7) furnishes an explicit formula for the symbol in terms of the operator and the orthonormal basis. The definition of Fourier coefficients tells us that for invariant operators we have

$$\sigma(j)_{mk} = (Te_j^k, e_j^m)_{L^2(M)}. \quad (2.8)$$

In particular, for the identity operator  $T = I$  we have  $\sigma_I(j) = I_{d_j}$ , where  $I_{d_j} \in \mathbb{C}^{d_j \times d_j}$  is the identity matrix.

Before proving Theorem 2.1, let us establish a formula relating symbols with respect to different orthonormal basis. If  $\{e_\alpha\}$  and  $\{f_\alpha\}$  are orthonormal bases of  $\mathcal{H}$ , we consider the unitary operator  $U$  determined by  $U(e_\alpha) = f_\alpha$ . Then we have

$$(Te_\alpha, e_\beta)_\mathcal{H} = (UTE_\alpha, Ue_\beta)_\mathcal{H} = (UTU^*Ue_\alpha, Ue_\beta)_\mathcal{H} = (UTU^*f_\alpha, f_\beta)_\mathcal{H}.$$

If  $(\sigma_T, \{e_\alpha\})$  denotes the symbol of  $T$  with respect to the orthonormal basis  $\{e_\alpha\}$  and  $(\sigma_{UTU^*}, \{f_\alpha\})$  denotes the symbol of  $UTU^*$  with respect to the orthonormal basis  $\{f_\alpha\}$  we have obtained the relation

$$(\sigma_T, \{e_\alpha\}) = (\sigma_{UTU^*}, \{f_\alpha\}). \quad (2.9)$$

Thus, the equivalence relation of basis  $\{e_\alpha\} \sim \{f_\alpha\}$  given by a unitary operator  $U$  induces the equivalence relation on the set  $\Sigma$  of symbols given by (2.9). In view of this, we can also think of the symbol as an element of the space  $\Sigma / \sim$  with the equivalence relation given by (2.9).

We make another remark concerning part (C) of Theorem 2.1. We use the condition that  $e_j^k$  are in the domain  $\text{Dom}(T^*)$  of  $T^*$  in showing the implication (B)  $\implies$  (C). Since  $e_j^k$ 's give a basis in  $\mathcal{H}$ , and are all contained in  $\text{Dom}(T^*)$ , it follows that  $\text{Dom}(T^*)$  is dense in  $\mathcal{H}$ . In particular, by [RS80, Theorem VIII.1],  $T$  must be closable (in

part (C)). These conditions are not restrictive for the further analysis since they are satisfied in the natural applications of this paper.

The principal application of the notions above will be as follows, except for in the sequel we will need more general operators  $E$  unbounded on  $\mathcal{H}$ . In order to distinguish from this general case, in the following theorem we use the notation  $E_o$ .

**Theorem 2.2.** *Continuing with the notation of Theorem 2.1, let  $E_o \in \mathcal{L}(\mathcal{H})$  be a linear continuous operator such that  $H_j$  are its eigenspaces:*

$$E_o e_j^k = \lambda_j e_j^k$$

for each  $j \in \mathbb{N}_0$  and all  $1 \leq k \leq d_j$ . Then equivalent conditions (A)–(C) imply the property

(E) For each  $j \in \mathbb{N}_0$  and  $1 \leq k \leq j$ , we have  $TE_o e_j^k = E_o T e_j^k$ , and if  $\lambda_j \neq \lambda_\ell$  for  $j \neq \ell$ , then (E) is equivalent to properties (A)–(C).

Moreover, if  $T$  extends to a bounded operator  $T \in \mathcal{L}(\mathcal{H})$  then equivalent properties (A)–(D) imply the condition

(F)  $TE_o = E_o T$  on  $\mathcal{H}$ ,

and if also  $\lambda_j \neq \lambda_\ell$  for  $j \neq \ell$ , then (F) is equivalent to (A)–(E).

For an operator  $T = F(E_o)$ , when it is well-defined by the spectral calculus, we have

$$\sigma_{F(E_o)}(j) = F(\lambda_j)I_{d_j}. \quad (2.10)$$

In fact, this is also well-defined then for a function  $F$  defined on  $\lambda_j$ , with finite values which are e.g.  $j$ -uniformly bounded (also for non self-adjoint  $E_o$ ). We first prove Theorem 2.1.

*Proof of Theorem 2.1.* (A)  $\implies$  (B). If  $T$  satisfies condition (A), we consider the matrix of  $T|_{H_j} : H_j \rightarrow H_j$  with respect to the orthonormal basis  $\{e_j^i : 1 \leq i \leq d_j\}$  of  $H_j$  and denote it by  $\beta(j)$ . Then

$$Te_j^k = \sum_{i=1}^{d_j} \beta(j)_{ki} e_j^i.$$

Consequently, we have

$$\widehat{Te_j^k}(\ell, m) = (Te_j^k, e_\ell^m) = \beta(j)_{km} \delta_{j\ell} = \beta(\ell)_{km} \delta_{j\ell}.$$

We take then  $\sigma(\ell) := \beta(\ell)^\top$ ; it belongs to  $\mathbb{C}^{d_\ell \times d_\ell}$  and satisfies (B).

(B)  $\implies$  (A). Since  $e_j^k \in \mathcal{H}^\infty$ , writing the series (2.3) for  $Te_j^k \in \mathcal{H}$ , we have

$$Te_j^k = \sum_{\ell} \sum_{m=1}^{d_\ell} \widehat{Te_j^k}(\ell, m) e_\ell^m = \sum_{\ell} \sum_{m=1}^{d_\ell} \sigma(\ell)_{mk} \delta_{j\ell} e_\ell^m = \sum_{m=1}^{d_j} \sigma(j)_{mk} e_j^m \in H_j. \quad (2.11)$$

Since  $\{e_j^m : 1 \leq m \leq d_j\}$  spans  $H_j$ , we obtain (A).

(B)  $\implies$  (C). We assume in addition that  $e_j^k$  are in the domain of  $T^*$  for all  $j$  and  $k$ . We also assume that for each  $\ell \in \mathbb{N}_0$  there exists a matrix  $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$  such that

$$\widehat{Te_j^k}(\ell, m) = \sigma(\ell)_{mk} \delta_{j\ell}. \quad (2.12)$$



Now, if  $f \in \mathcal{H}^\infty$ , then  $Tf \in \mathcal{H}$ , and by the inversion formula (2.3) we have

$$f = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \widehat{f}(j, k) e_j^k.$$

Now, using this and the fact that all  $e_\ell^m$  are in the domain of  $T^*$ , we have

$$\begin{aligned} \widehat{Tf}(\ell, m) &= (Tf, e_\ell^m) \\ &= (f, T^* e_\ell^m) \\ &= \left( \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \widehat{f}(j, k) e_j^k, T^* e_\ell^m \right) \\ &= \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \widehat{f}(j, k) (Te_j^k, e_\ell^m) \\ &= \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \widehat{f}(j, k) \widehat{Te_j^k}(\ell, m) \\ &= \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \widehat{f}(j, k) \sigma(\ell)_{mk} \delta_{j\ell} \\ &= \sum_{k=1}^{d_j} \sigma(\ell)_{mk} \widehat{f}(\ell, k), \end{aligned}$$

where we also used (2.12). Hence  $\widehat{Tf}(\ell) = \sigma(\ell) \widehat{f}(\ell)$ , yielding (C).

(C)  $\implies$  (B). If  $\widehat{Tf}(\ell) = \sigma(\ell) \widehat{f}(\ell)$ , then

$$\widehat{Te_j^k}(\ell, m) = \left( \sigma(\ell) \widehat{e_j^k}(\ell) \right)_m = \sum_{i=1}^{d_j} \sigma(\ell)_{mi} \widehat{e_j^k}(\ell, i) = \sum_{i=1}^{d_j} \sigma(\ell)_{mi} \delta_{j\ell} \delta_{ki} = \sigma(\ell)_{mk} \delta_{j\ell},$$

which gives (B), even without any assumptions on  $T^*$ .

(D)  $\implies$  (A). We take  $f \in H_j$ . Then  $P_j f \in H_j$  since  $P_j f = f$ , so that by assumption (D) we have

$$Tf = TP_j f = P_j Tf \in H_j,$$

implying (A).

(A)  $\implies$  (D). For this part we assume in addition that  $T$  extends to a bounded operator  $T \in \mathcal{L}(\mathcal{H})$ . First, we show that this together with (A) implies that  $T(H_j^\perp)$  is orthogonal to  $H_j$ . For  $g \in H_j^\perp$ , we can write

$$g = \sum_{\ell \neq j} \sum_{k=1}^{d_\ell} (g, e_\ell^k) e_\ell^k$$

with the convergence in  $\mathcal{H}$ , so that

$$Tg = \sum_{\ell \neq j} \sum_{k=1}^{d_\ell} (g, e_\ell^k) T e_\ell^k$$

with the convergence in  $\mathcal{H}$  due to the boundedness of  $T$  on  $\mathcal{H}$ . Since by (A) we have  $T e_\ell^k \in H_\ell \subset H_j^\perp$  for  $\ell \neq j$  we conclude that  $Tg$  is orthogonal to  $H_j$ .

Let now  $f \in \mathcal{H}^\infty$ . Writing  $f = f_1 + f_2$  with  $f_1 := P_j f$  so that  $f_1 \in H_j$  and  $f_2 \in H_j^\perp$  are both in  $\mathcal{H}^\infty$ , we have

$$P_j T f = P_j T f_1 + P_j T f_2 = T f_1 = T P_j f,$$

since the proved claim  $P_j f_2 = 0$  implies that  $P_j T f_2 = 0$ .  $\square$

We now continue with the proof of Theorem 2.2 when the basis  $e_j^k$  corresponds to the eigenvectors of an operator  $E_o \in \mathcal{L}(\mathcal{H})$ .

*Proof of Theorem 2.2.* (A)  $\implies$  (E). Let us fix some  $e_j^k$ . By condition (A) we can write

$$T e_j^k = \sum_{i=1}^{d_j} \alpha_i e_j^i$$

for some constants  $\alpha_i$ . Then

$$E_o T e_j^k = E_o \sum_{i=1}^{d_j} \alpha_i e_j^i = \sum_{i=1}^{d_j} \alpha_i \lambda_j e_j^i = \lambda_j \sum_{i=1}^{d_j} \alpha_i e_j^i = \lambda_j T e_j^k = T \lambda_j e_j^k = T E_o e_j^k,$$

which shows (E).

(E)  $\implies$  (A). We note that it is enough to prove that  $T e_j^k \in H_j$  since  $\{e_j^k : 1 \leq k \leq d_j\}$  forms a basis of the finite dimensional space  $H_j$ . We can assume that  $T e_j^k \neq 0$  since otherwise there is nothing to prove. We recall that  $E_o e_j^k = \lambda_j e_j^k$ . Using property (E), we have

$$\lambda_j T e_j^k = T E_o e_j^k = E_o T e_j^k.$$

Hence  $T e_j^k \in \mathcal{H}$  is a non-zero eigenvector of  $E_o$  corresponding to the eigenvalue  $\lambda_j$ . Consequently, since  $H_j$  are maximal eigenspaces corresponding to  $\lambda_j$ , we must have  $T e_j^k \in H_j$ .

(E)  $\implies$  (F). Since we have already shown that (A)–(C) always imply (E), it is enough to prove that (E) implies (F) under the additional assumption that  $T \in \mathcal{L}(\mathcal{H})$ .

Let us write  $S := E_o \circ T$ ,  $D := T \circ E_o$  and let  $f \in \mathcal{H}$ . Under the assumptions both  $S$  and  $D$  are bounded on  $\mathcal{H}$ , and hence the formula (2.3) implies

$$Sf = \lim_N \sum_{j=0}^N \sum_{k=1}^{d_j} (f, e_j^k) S e_j^k = \lim_N \sum_{j=0}^N \sum_{k=1}^{d_j} (f, e_j^k) D e_j^k = Df,$$

with the convergent series in  $\mathcal{H}$ .

(F)  $\implies$  (A). We note that we require  $T \in \mathcal{L}(\mathcal{H})$  in order for  $T E_o$  and  $E_o T$  to make sense on  $\mathcal{H}$ . It is clear that (F) implies (E), and under the additional assumption

that  $\lambda_j \neq \lambda_\ell$  for  $j \neq \ell$  we already know that (A)–(C) and (E) are equivalent. If  $T$  is bounded on  $\mathcal{H}$ , then they are also equivalent to (D).  $\square$

We have the following criterion for the extendability of a densely defined invariant operator  $T : \mathcal{H}^\infty \rightarrow \mathcal{H}$  to  $\mathcal{L}(\mathcal{H})$ , which was an additional hypothesis for properties (D) and (F). In the statements below we fix a partition into  $H_j$ 's as in (2.1) and the invariance refers to it.

**Theorem 2.3.** *An invariant linear operator  $T : \mathcal{H}^\infty \rightarrow \mathcal{H}$  extends to a bounded operator from  $\mathcal{H}$  to  $\mathcal{H}$  if and only if its symbol  $\sigma$  satisfies  $\sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{\mathcal{L}(H_\ell)} < \infty$ .*

Moreover, denoting this extension also by  $T$ , we have

$$\|T\|_{\mathcal{L}(\mathcal{H})} = \sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{\mathcal{L}(H_\ell)}.$$

*Proof.* We will often abbreviate writing  $\|\sigma(\ell)\|_{op} := \|\sigma(\ell)\|_{\mathcal{L}(H_\ell)}$ . Let us first suppose that  $\|\sigma(\ell)\|_{op} \leq C$  for all  $\ell \in \mathbb{N}_0$ . By the Plancherel formula (2.4) we have

$$\begin{aligned} \|Tf\|_{\mathcal{H}}^2 &= \|\widehat{Tf}\|_{\ell^2(\mathbb{N}_0, \Sigma)}^2 \\ &= \sum_{\ell} \|\widehat{Tf}(\ell)\|_{\ell^2(\mathbb{C}^{d_\ell})}^2 \\ &= \sum_{\ell} \|\sigma(\ell)\widehat{f}(\ell)\|_{\ell^2(\mathbb{C}^{d_\ell})}^2 \\ &\leq \sum_{\ell} \|\sigma(\ell)\|_{op}^2 \|\widehat{f}(\ell)\|_{\ell^2(\mathbb{C}^{d_\ell})}^2 \\ &\leq \sup_{\ell} \|\sigma(\ell)\|_{op}^2 \sum_{\ell} \|\widehat{f}(\ell)\|_{\ell^2(\mathbb{C}^{d_\ell})}^2 \\ &= \left( \sup_{\ell} \|\sigma(\ell)\|_{op} \right)^2 \|f\|_{\mathcal{H}}^2. \end{aligned}$$

Conversely, let us suppose that  $T$  is bounded on  $\mathcal{H}$ . Then there exists a constant  $C > 0$  such that  $\|Tf\|_{\mathcal{H}} \leq C$  for all  $f$  such that  $\|f\|_{\mathcal{H}} = 1$ . We can take  $C := \|T\|_{\mathcal{L}(\mathcal{H})}$ . Hence

$$T|_{H_j} : H_j \rightarrow H_j$$

is bounded and  $\|T|_{H_j}\|_{\mathcal{L}(H_j)} \leq C$ . On the other hand, let  $\beta(j)$  denote the matrix of  $T|_{H_j} : H_j \rightarrow H_j$  with respect to the orthonormal basis  $\{e_j^i : 1 \leq i \leq d_j\}$  of  $H_j$  as in the proof of Part (A) implies (B) in Theorem 2.1. We consider an unitary operator  $U : H_j \rightarrow \mathbb{C}^{d_j}$  which defines coordinates in  $\mathbb{C}^{d_j}$  of vectors in  $H_j$  with respect to the orthonormal basis  $\{e_j^k : 1 \leq k \leq d_j\}$  of  $H_j$ . We also consider the operator  $A(j) : \mathbb{C}^{d_j} \rightarrow \mathbb{C}^{d_j}$  induced by the matrix  $\beta(j)$ . Then

$$T|_{H_j} = U^* A(j) U,$$

and

$$\|\sigma(j)\|_{op} = \|\beta(j)\|_{op} = \|A(j)\|_{op} = \|T|_{H_j}\|_{\mathcal{L}(H_j)} \leq C,$$

completing the proof.  $\square$

We also record the formula for the symbol of the composition of two invariant operators:

**Proposition 2.4.** *If  $S, T : \mathcal{H}^\infty \rightarrow \mathcal{H}$  are invariant operators with respect to the same orthonormal partition, and such that the domain of  $S \circ T$  contains  $\mathcal{H}^\infty$ , then  $S \circ T : \mathcal{H}^\infty \rightarrow \mathcal{H}$  is also invariant with respect to the same partition. Moreover, if  $\sigma_S$  denotes the symbol of  $S$  and  $\sigma_T$  denotes the symbols of  $T$  with respect to the same orthonormal basis then*

$$\sigma_{S \circ T} = \sigma_S \sigma_T,$$

*i.e.  $\sigma_{S \circ T}(j) = \sigma_S(j)\sigma_T(j)$  for all  $j \in \mathbb{N}_0$ .*

*Proof.* Recalling the definition of the composition of densely defined operators, the domain of  $S \circ T$  is the space of functions  $f$  in the domain of  $T$  such that  $Tf$  is in the domain of  $S$ , in which case we set  $(S \circ T)f = S(Tf)$ . The assumption says that we are in the position to use Theorem 2.1. Applying the condition (C) of Theorem 2.1 repeatedly, we have

$$(\widehat{S \circ T}f)(j) = \widehat{S}(Tf)(j) = \sigma_S(j)\widehat{T}f(j) = \sigma_S(j)\sigma_T(j)\widehat{f}(j),$$

so that  $S \circ T$  is invariant by Part (C) of Theorem 2.1.  $\square$

We now show another application of the above notions to give a characterisation of Schatten classes of invariant operators in terms of their symbols.

**Theorem 2.5.** *Let  $0 < r < \infty$ . An invariant operator  $T \in \mathcal{L}(\mathcal{H})$  with symbol  $\sigma$  is in the Schatten class  $S_r(\mathcal{H})$  if and only if*

$$\sum_{\ell=0}^{\infty} \|\sigma(\ell)\|_{S_r(H_\ell)}^r < \infty.$$

Moreover

$$\|T\|_{S_r(\mathcal{H})} = \left( \sum_{\ell=0}^{\infty} \|\sigma(\ell)\|_{S_r(H_\ell)}^r \right)^{1/r}. \quad (2.13)$$

*In particular, if  $T$  is in the trace class  $S_1(\mathcal{H})$ , then we have the trace formula*

$$\mathrm{Tr}(T) = \sum_{\ell=0}^{\infty} \mathrm{Tr}(\sigma(\ell)). \quad (2.14)$$

*Proof.* First, we claim that Schatten classes of invariant operators can be characterised in terms of the projections to the eigenspaces  $H_\ell$ :

$$\|T\|_{S_r(\mathcal{H})}^r = \sum_{\ell=0}^{\infty} \|T|_{H_\ell}\|_{S_r(H_\ell)}^r. \quad (2.15)$$

Let us prove (2.15). Since

$$\|T\|_{S_r} = \||T|\|_{S_r}$$

we can assume without loss of generality that  $T$  is positive definite. We first observe that  $\lambda$  is an eigenvalue (singular value) of  $T$  if and only if  $\lambda$  is an eigenvalue (singular value) of  $T|_{H_{\ell(\lambda)}}$  for some  $\ell(\lambda)$ . Indeed, if  $\lambda$  is an eigenvalue of  $T$  there exists  $\varphi_\lambda \in \mathcal{H} \setminus \{0\}$  such that  $T\varphi_\lambda = \lambda\varphi_\lambda$ . Using Part (D) of Theorem 2.1, we get that

$$TP_\ell\varphi_\lambda = \lambda P_\ell\varphi_\lambda$$

holds for every  $\ell$ . Since  $\varphi_\lambda \neq 0$ , there exists  $\ell(\lambda)$  such that  $P_{\ell(\lambda)}\varphi_\lambda \neq 0$ . Consequently,  $\lambda$  is the eigenvalue of  $T|_{H_{\ell(\lambda)}} = TP_{\ell(\lambda)}$ . Conversely, since  $T(H_{\ell(\lambda)}) \subset H_{\ell(\lambda)}$ , an eigenvalue of  $T|_{H_{\ell(\lambda)}}$  is also an eigenvalue of  $T$ . Therefore, we obtain (2.15).

Now, given (2.15), to prove (2.13), it is enough to check that

$$\|\sigma(\ell)\|_{S_r(H_\ell)} = \|T|_{H_\ell}\|_{S_r(H_\ell)}. \quad (2.16)$$

To prove (2.16) we consider an unitary operator  $U : H_\ell \rightarrow \mathbb{C}^{d_\ell}$  which defines coordinates in  $\mathbb{C}^{d_\ell}$  of functions in  $H_\ell$  with respect to the orthonormal basis  $\{e_\ell^k : 1 \leq k \leq d_\ell\}$  of  $H_\ell$ . We also consider the operator  $A(\ell) : \mathbb{C}^{d_\ell} \rightarrow \mathbb{C}^{d_\ell}$  induced by the matrix  $(\sigma_T(\ell))^\top$ . Then

$$T|_{H_\ell} = U^* A(\ell) U,$$

and basic properties of Schatten quasinorms imply that

$$\|T|_{H_\ell}\|_{S_r(H_\ell)} = \|A(\ell)\|_{S_r(\mathbb{C}^{d_\ell})} = \|\sigma(\ell)\|_{S_r},$$

completing the proof of (2.16) and of (2.13).

Finally, let us prove (2.14) for operators in the trace class  $S_1(\mathcal{H})$ . Since the trace  $\text{Tr}(T)$  does not depend on the choice of the orthonormal basis in  $\mathcal{H}$ , using property (C) and formula (2.11), we can write

$$\begin{aligned} \text{Tr}(T) &= \sum_{\ell} \sum_{k=1}^{d_\ell} (T e_\ell^k, e_\ell^k) = \sum_{\ell} \sum_{k=1}^{d_\ell} \sum_{m=1}^{d_\ell} \sigma(\ell)_{mk} (e_\ell^m, e_\ell^k) \\ &= \sum_{\ell} \sum_{k=1}^{d_\ell} \sum_{m=1}^{d_\ell} \sigma(\ell)_{mk} \delta_{mk} = \sum_{\ell} \sum_{k=1}^{d_\ell} \sigma(\ell)_{kk} = \sum_{\ell} \text{Tr}(\sigma(\ell)), \end{aligned}$$

completing the proof.  $\square$

*Remark 2.6.* We note that the membership in  $\mathcal{L}(\mathcal{H})$  and in the Schatten classes  $S_r(\mathcal{H})$  does not depend on the decomposition of  $\mathcal{H}$  into subspaces  $H_j$  as in (2.1). However, the notion of invariance does depend on it. For example, let  $\mathcal{H} = L^2(\mathbb{T}^n)$  for the  $n$ -torus  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ . Choosing

$$H_j = \text{span}\{e^{2\pi i j \cdot x}\}, \quad j \in \mathbb{Z}^n,$$

we recover the construction of Section 6 on compact Lie groups and moreover, invariant operators with respect to  $\{H_j\}_{j \in \mathbb{Z}^n}$  are the translation invariant operators on the torus  $\mathbb{T}^n$ . However, to recover the construction of Section 4 on manifolds, we take  $\widetilde{H}_\ell$  to be the eigenspaces of the Laplacian  $E$  on  $\mathbb{T}^n$ , so that

$$\widetilde{H}_\ell = \bigoplus_{|j|^2=\ell} H_j = \text{span}\{e^{2\pi i j \cdot x} : j \in \mathbb{Z}^n \text{ and } |j|^2 = \ell\}, \quad \ell \in \mathbb{N}_0.$$

Then translation invariant operators on  $\mathbb{T}^n$ , i.e. operators invariant relative to the partition  $\{H_j\}_{j \in \mathbb{Z}^n}$ , are also invariant relative to the partition  $\{\widetilde{H}_\ell\}_{\ell \in \mathbb{N}_0}$  (or relative to the Laplacian, in terminology of Section 4). If we have information on the eigenvalues of  $E$ , like we do on the torus, we may sometimes also recover invariant operators relative to the partition  $\{\widetilde{H}_\ell\}_{\ell \in \mathbb{N}_0}$  as linear combinations of translation invariant operators composed with phase shifts and complex conjugation.

### 3. FOURIER ANALYSIS ASSOCIATED TO AN ELLIPTIC OPERATOR

Our main application will be to study operators on compact manifolds, so we start this section by describing the discrete Fourier series associated to an elliptic positive pseudo-differential operator as an adaptation of the construction in Section 2. In order to fix the notation for the rest of the paper we may give some explicit expressions for notions of Section 2 in the present setting.

Let  $M$  be a compact smooth manifold of dimension  $n$  without boundary, endowed with a fixed volume  $dx$ . We denote by  $\Psi^\nu(M)$  the Hörmander class of pseudo-differential operators of order  $\nu \in \mathbb{R}$ , i.e. operators which, in every coordinate chart, are operators in Hörmander classes on  $\mathbb{R}^n$  with symbols in  $S'_{1,0}$ , see e.g. [Shu01] or [RT10]. In this paper we will be using the class  $\Psi_{cl}^\nu(M)$  of classical operators, i.e. operators with symbols having (in all local coordinates) an asymptotic expansion of the symbol in positively homogeneous components (see e.g. [Dui11]). Furthermore, we denote by  $\Psi_+^\nu(M)$  the class of positive definite operators in  $\Psi_{cl}^\nu(M)$ , and by  $\Psi_e^\nu(M)$  the class of elliptic operators in  $\Psi_{cl}^\nu(M)$ . Finally,

$$\Psi_{+e}^\nu(M) := \Psi_+^\nu(M) \cap \Psi_e^\nu(M)$$

will denote the class of classical positive elliptic pseudo-differential operators of order  $\nu$ . We note that complex powers of such operators are well-defined, see e.g. Seeley [See67]. In fact, all pseudo-differential operators considered in this paper will be classical, so we may omit explicitly mentioning it every time, but we note that we could equally work with general operators in  $\Psi^\nu(M)$  since their powers have similar properties, see e.g. [Str72].

We now associate a discrete Fourier analysis to the operator  $E \in \Psi_{+e}^\nu(M)$  inspired by those constructions considered by Seeley ([See65], [See69]), see also Greenfield and Wallach [GW73]. However, we adapt it to our purposes and in the sequel also prove several auxiliary statements concerning the eigenvalues of  $E$  and their multiplicities, useful to us in the subsequent analysis. In general, the construction below is exactly the one appearing in Theorem 2.1.

The eigenvalues of  $E$  (counted without multiplicities) form a sequence  $\{\lambda_j\}$  which we order so that

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots. \quad (3.1)$$

For each eigenvalue  $\lambda_j$ , there is the corresponding finite dimensional eigenspace  $H_j$  of functions on  $M$ , which are smooth due to the ellipticity of  $E$ . We set

$$d_j := \dim H_j, \text{ and } H_0 := \ker E, \lambda_0 := 0.$$

We also set  $d_0 := \dim H_0$ . Since the operator  $E$  is elliptic, it is Fredholm, hence also  $d_0 < \infty$  (we can refer to [Ati68], [Hör85a] for various properties of  $H_0$  and  $d_0$ ).

We fix an orthonormal basis of  $L^2(M)$  consisting of eigenfunctions of  $E$ :

$$\{e_j^k\}_{j \geq 0}^{1 \leq k \leq d_j}, \quad (3.2)$$

where  $\{e_j^k\}_{1 \leq k \leq d_j}$  is an orthonormal basis of  $H_j$ . Let  $P_j : L^2(M) \rightarrow H_j$  be the corresponding projection. We shall denote by  $(\cdot, \cdot)$  the inner product of  $L^2(M)$ . We

observe that we have

$$P_j f = \sum_{k=1}^{d_j} (f, e_j^k) e_j^k,$$

for  $f \in L^2(M)$ . The ‘Fourier’ series takes the form

$$f = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (f, e_j^k) e_j^k,$$

for each  $f \in L^2(M)$ . The Fourier coefficients of  $f \in L^2(M)$  with respect to the orthonormal basis  $\{e_j^k\}$  will be denoted by

$$(\mathcal{F}f)(j, k) := \widehat{f}(j, k) := (f, e_j^k). \quad (3.3)$$

We will call the collection of  $\widehat{f}(j, k)$  the *Fourier coefficients of  $f$  relative to  $E$* , or simply the *Fourier coefficients of  $f$* .

Since  $\{e_j^k\}_{j \geq 0}^{1 \leq k \leq d_j}$  forms a complete orthonormal system in  $L^2(M)$ , for all  $f \in L^2(M)$  we have the Plancherel formula (2.4), namely,

$$\|f\|_{L^2(M)}^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |(f, e_j^k)|^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} |\widehat{f}(j, k)|^2 = \|\widehat{f}\|_{\ell^2(\mathbb{N}_0, \Sigma)}^2, \quad (3.4)$$

where the space  $\ell^2(\mathbb{N}_0, \Sigma)$  and its norm are as in (2.5) and (2.6).

We can think of  $\mathcal{F} = \mathcal{F}_M$  as of the Fourier transform being an isometry from  $L^2(M)$  into  $\ell^2(\mathbb{N}_0, \Sigma)$ . The inverse of this Fourier transform can be then expressed by

$$(\mathcal{F}^{-1}h)(x) = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} h(j, k) e_j^k(x). \quad (3.5)$$

If  $f \in L^2(M)$ , we also write

$$\widehat{f}(j) = \begin{pmatrix} \widehat{f}(j, 1) \\ \vdots \\ \widehat{f}(j, d_j) \end{pmatrix} \in \mathbb{C}^{d_j},$$

thus thinking of the Fourier transform always as a column vector. In particular, we think of

$$\widehat{e}_j^k(\ell) = \left( \widehat{e}_j^k(\ell, m) \right)_{m=1}^{d_\ell}$$

as of a column, and we notice that

$$\widehat{e}_j^k(\ell, m) = \delta_{j\ell} \delta_{km}. \quad (3.6)$$

Smooth functions on  $M$  can be characterised by

$$\begin{aligned} f \in C^\infty(M) &\iff \forall N \exists C_N : |\widehat{f}(j, k)| \leq C_N (1 + \lambda_j)^{-N} \text{ for all } j, k \\ &\iff \forall N \exists C_N : |\widehat{f}(j)| \leq C_N (1 + \lambda_j)^{-N} \text{ for all } j, \end{aligned} \quad (3.7)$$

where  $|\widehat{f}(j)|$  is the norm of the vector  $\widehat{f}(j) \in \mathbb{C}^{d_j}$ . The implication ' $\Leftarrow$ ' here is immediate, while ' $\Rightarrow$ ' follows from the Plancherel formula (2.4) and the fact that for  $f \in C^\infty(M)$  we have  $(I + E)^N f \in L^2(M)$  for any  $N$ .

For  $u \in \mathcal{D}'(M)$ , we denote its Fourier coefficient

$$\widehat{u}(j, k) := u(\overline{e_j^k}),$$

and by duality, the space of distributions can be characterised by

$$f \in \mathcal{D}'(M) \iff \exists M \exists C : |\widehat{u}(j, k)| \leq C(1 + \lambda_j)^M \text{ for all } j, k.$$

We will denote by  $H^s(M)$  the usual Sobolev space over  $L^2$  on  $M$ . This space can be defined in local coordinates or, by the fact that  $E \in \Psi_{+e}^\nu(M)$  is positive and elliptic with  $\nu > 0$ , it can be characterised by

$$\begin{aligned} f \in H^s(M) &\iff (I + E)^{s/\nu} f \in L^2(M) \iff \{(1 + \lambda_j)^{s/\nu} \widehat{f}(j)\}_j \in \ell^2(\mathbb{N}_0, \Sigma) \\ &\iff \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (1 + \lambda_j)^{2s/\nu} |\widehat{f}(j, k)|^2 < \infty. \end{aligned} \quad (3.8)$$

the last equivalence following from the Plancherel formula (2.4). For the characterisation of analytic functions (on compact manifolds  $M$ ) we refer to Seeley [See69].

#### 4. INVARIANT OPERATORS AND SYMBOLS ON COMPACT MANIFOLDS

We now discuss an application of a notion of an invariant operator and of its symbol from Theorem 2.1 in the case of  $\mathcal{H} = L^2(M)$  and  $\mathcal{H}^\infty = C^\infty(M)$  and describe its basic properties. We will consider operators  $T$  densely defined on  $L^2(M)$ , and we will be making a natural assumption that their domain contains  $C^\infty(M)$ . We also note that while in Theorem 2.2 it was assumed that the operator  $E_o$  is bounded on  $\mathcal{H}$ , this is no longer the case for the operator  $E$  here. Indeed, an elliptic pseudo-differential operator  $E \in \Psi_{+e}^\nu(M)$  of order  $\nu > 0$  is not bounded on  $L^2(M)$ .

Moreover, we do not want to assume that  $T$  extends to a bounded operator on  $L^2(M)$  to obtain analogues of properties (D) and (F) in Section 2, because this is too restrictive from the point of view of differential operators. Instead, we show that in the present setting it is enough to assume that  $T$  extends to a continuous operator on  $\mathcal{D}'(M)$  to reach the same conclusions.

So, we combine the statement of Theorem 2.1 and the necessary modification of Theorem 2.2 to the setting of Section 3 as follows.

We also remark that Part (iv) of the following theorem provides a correct formulation for a missing assumption in [DR14a, Theorem 3.1, (iv)].

**Theorem 4.1.** *Let  $M$  be a closed manifold and let  $T : C^\infty(M) \rightarrow L^2(M)$  be a linear operator. Then the following conditions are equivalent:*

- (i) *For each  $j \in \mathbb{N}_0$ , we have  $T(H_j) \subset H_j$ .*
- (ii) *For each  $j \in \mathbb{N}_0$  and  $1 \leq k \leq j$ , we have  $TEe_j^k = ETe_j^k$ .*
- (iii) *For each  $\ell \in \mathbb{N}_0$  there exists a matrix  $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$  such that for all  $e_j^k$*

$$\widehat{Te_j^k}(\ell, m) = \sigma(\ell)_{mk} \delta_{j\ell}. \quad (4.1)$$



- (iv) If, in addition, the domain of  $T^*$  contains  $C^\infty(M)$ , then for each  $\ell \in \mathbb{N}_0$  there exists a matrix  $\sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}$  such that

$$\widehat{Tf}(\ell) = \sigma(\ell)\widehat{f}(\ell)$$

for all  $f \in C^\infty(M)$ .

The matrices  $\sigma(\ell)$  in (iii) and (iv) coincide.

If  $T$  extends to a linear continuous operator  $T : \mathcal{D}'(M) \rightarrow \mathcal{D}'(M)$  then the above properties are also equivalent to the following ones:

- (v) For each  $j \in \mathbb{N}_0$ , we have  $TP_j = P_jT$  on  $C^\infty(M)$ .  
 (vi)  $TE = ET$  on  $L^2(M)$ .

If any of the equivalent conditions (i)–(iv) of Theorem 4.1 are satisfied, we say that the operator  $T : C^\infty(M) \rightarrow L^2(M)$  is *invariant* (or is a *Fourier multiplier*) *relative to  $E$* . We can also say that  $T$  is  *$E$ -invariant* or is an  *$E$ -multiplier*. This recovers the notion of invariant operators given by Theorem 2.1, with respect to the partitions  $H_j$ 's in (2.1) which are fixed being the eigenspaces of  $E$ . When there is no risk of confusion we will just refer to such kind of operators as invariant operators or as multipliers. It is clear from (i) that the operator  $E$  itself or functions of  $E$  defined by the functional calculus are invariant relative to  $E$ .

We note that the boundedness of  $T$  on  $L^2(M)$  needed for conditions (D) and (F) in Theorem 2.1 and in Theorem 2.2 is now replaced by the condition that  $T$  is continuous on  $\mathcal{D}'(M)$  which explored the additional structure of  $L^2(M)$  and allows application to differential operators.

We call  $\sigma$  in (iii) and (iv) the *matrix symbol* of  $T$  or simply the *symbol*. It is an element of the space  $\Sigma = \Sigma_M$  defined by

$$\Sigma_M := \{\sigma : \mathbb{N}_0 \ni \ell \mapsto \sigma(\ell) \in \mathbb{C}^{d_\ell \times d_\ell}\}. \quad (4.2)$$

Since the expression for the symbol depends only on the basis  $e_j^k$  and not on the operator  $E$  itself, this notion coincides with the symbol defined in Theorem 2.1.

Let us comment on several conditions in Theorem 4.1 in this setting. Assumptions (v) and (vi) are stronger than those in (i)–(iv). On one hand, clearly (vi) contains (ii). On the other hand, as we will see in the proof, assumption (v) implies (i) without the additional hypothesis that  $T$  is continuous on  $\mathcal{D}'(M)$ .

In analogy to the strong commutativity in (v), if  $T$  is continuous on  $\mathcal{D}'(M)$ , so that all the assumptions (i)–(vi) are equivalent, we may say that  $T$  is *strongly invariant relative to  $E$*  in this case.

The expressions in (vi) make sense as both sides are defined (and even continuous) on  $\mathcal{D}'(M)$ .

We also note that without additional assumptions, it is known from the general theory of densely defined operators on Hilbert spaces that conditions (v) and (vi) are generally not equivalent, see e.g. Reed and Simon [RS80, Section VIII.5]. If  $T$  is a differential operator, the additional assumption of continuity on  $\mathcal{D}'(M)$  for parts (v) and (vi) is satisfied. In [GW73, Section 1, Definition 1] Greenfield and Wallach called a differential operator  $D$  to be an  *$E$ -invariant operator* if  $ED = DE$ , which is our condition (vi). However, Theorem 4.1 describes more general operators as well as reformulates them in the form of Fourier multipliers that will be explored in the sequel.

There will be several useful classes of symbols, in particular the moderate growth class

$$\mathcal{S}'(\Sigma) := \{\sigma \in \Sigma : \exists N, C \text{ such that } \|\sigma(\ell)\|_{op} \leq C(1 + \lambda_\ell)^N \forall \ell \in \mathbb{N}_0\}, \quad (4.3)$$

where

$$\|\sigma(\ell)\|_{op} = \|\sigma(\ell)\|_{\mathcal{L}(H_\ell)}$$

denotes the matrix multiplication operator norm with respect to  $\ell^2(\mathbb{C}^{d_\ell})$ .

In the case when  $M$  is a compact Lie group and  $E$  is a Laplacian on  $G$ , left-invariant operators on  $G$ , i.e. operators commuting with the left action of  $G$ , are also invariant relative to  $E$  in the sense of Theorem 4.1; this will be shown in Proposition 6.1 after we investigate in Section 6 the relation between the symbol in Theorem 4.1 and matrix symbols of operators on compact Lie groups. However, we need an adaptation of the above construction since the natural decomposition into  $H_j$ 's in (2.1) may in general violate the condition (3.1).

As in Section 2 since the notion of the symbol depends only on the basis, for the identity operator  $T = I$  we have

$$\sigma_I(j) = I_{d_j},$$

where  $I_{d_j} \in \mathbb{C}^{I_{d_j} \times I_{d_j}}$  is the identity matrix, and for an operator  $T = F(E)$ , when it is well-defined by the spectral calculus, we have

$$\sigma_{F(E)}(j) = F(\lambda_j)I_{d_j}. \quad (4.4)$$

*Proof of Theorem 4.1.* Once the basis  $e_j^k$  is fixed, the equivalence of (i), (ii) and (iv) follows from the equivalence of (A), (B) and (C) in Theorem 2.1.

(ii)  $\implies$  (i). We first note that both  $ET$  and  $TE$  are well-defined on  $e_j^k$ : for the former, since  $e_j^k$  is smooth, we have  $Te_j^k \in L^2(M)$  and hence in  $\mathcal{D}'(M)$  where  $E$  is well-defined as a pseudo-differential operator, while, for the latter,  $Ee_j^k = \lambda_j e_j^k \in H_j \subset C^\infty(M)$  and hence it is in the domain of  $T$ . The rest of the proof is identical to (E)  $\implies$  (A) in the proof of Theorem 2.2.

(i)  $\implies$  (ii). This is the same as (A)  $\implies$  (E) in the proof of Theorem 2.2.

(v)  $\implies$  (i). We take  $f \in H_j$ . Then  $P_j f = f \in C^\infty(M)$  so that by assumption (v) we have

$$Tf = TP_j f = P_j Tf \in H_j,$$

implying (i).

(i)  $\implies$  (v). We now assume in addition that  $T$  is continuous on  $\mathcal{D}'(M)$ . First, we show that (i) implies that for any  $g \in H_j^\perp \subset L^2(M)$ , we have  $\langle Tg, \overline{e_j^k} \rangle = 0$  in the sense of distributions. We can write

$$g = \sum_{\ell \neq j} \sum_{k=1}^{d_\ell} (g, e_\ell^k) e_\ell^k$$

with the convergence in  $L^2(M)$ . Hence

$$Tg = \sum_{\ell \neq j} \sum_{k=1}^{d_\ell} (g, e_\ell^k) T e_\ell^k$$

with the convergence in  $\mathcal{D}'(M)$ . Since  $Te_\ell^k \in H_\ell \subset H_j^\perp$  for  $\ell \neq j$  we conclude that  $Tg$  is orthogonal to  $H_j$ .

Let now  $f \in C^\infty(M)$ . Writing  $f = f_1 + f_2$  with  $f_1 = P_j f$  so that  $f_1 \in H_j$  and  $f_2 \in H_j^\perp$  are necessarily smooth, and  $P_j f_2 = 0$ , we have

$$P_j T f = P_j T f_1 + P_j T f_2 = T f_1 = T P_j f,$$

since the above property implies that  $P_j T f_2 = 0$ .

(vi)  $\implies$  (ii). Trivial.

(ii)  $\implies$  (vi). For the following, we assume that  $T$  is continuous on  $\mathcal{D}'(M)$ . Let us write  $S := E \circ T$ ,  $D := T \circ E$  and let  $f \in L^2(M)$ . We can write

$$f = \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} (f, e_j^k) e_j^k$$

with the series convergent in  $L^2(M)$ . Since both  $S$  and  $D$  are continuous on  $\mathcal{D}'(M)$ , we now have

$$Sf = \lim_N \sum_{j=0}^N \sum_{k=1}^{d_j} (f, e_j^k) S e_j^k = \lim_N \sum_{j=0}^N \sum_{k=1}^{d_j} (f, e_j^k) D e_j^k = Df.$$

The limit should be understood in  $\mathcal{D}'(M)$ . Indeed, if we write

$$f_N = \sum_{j=0}^N \sum_{k=1}^{d_j} (f, e_j^k) e_j^k,$$

then  $f_N \rightarrow f$  in  $L^2$  and hence also in  $\mathcal{D}'(M)$ , which implies  $Sf_N \rightarrow Sf$  and  $Df_N \rightarrow Df$  in  $\mathcal{D}'(M)$ .  $\square$

We now discuss how invariant operators can be expressed in terms of their symbols.

**Proposition 4.2.** *An invariant operator  $T_\sigma$  associated to the symbol  $\sigma$  can be written in the following way:*

$$\begin{aligned} T_\sigma f(x) &= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} (\sigma(\ell) \widehat{f}(\ell))_m e_\ell^m(x) \\ &= \sum_{\ell=0}^{\infty} [\sigma(\ell) \widehat{f}(\ell)]^\top e_\ell(x), \end{aligned} \tag{4.5}$$

where  $[\sigma(\ell) \widehat{f}(\ell)]$  denotes the column-vector, and  $[\sigma(\ell) \widehat{f}(\ell)]^\top e_\ell(x)$  denotes the multiplication (the scalar product) of the column-vector  $[\sigma(\ell) \widehat{f}(\ell)]$  with the column-vector  $e_\ell(x) = (e_\ell^1(x), \dots, e_\ell^{d_\ell}(x))^\top$ . In particular, we also have

$$(T_\sigma e_j^k)(x) = \sum_{m=1}^{d_j} \sigma(j)_{mk} e_j^m(x). \tag{4.6}$$

If  $\sigma \in \mathcal{S}'(\Sigma)$  and  $f \in C^\infty(M)$ , the convergence in (4.5) is uniform.

*Proof.* Formula (4.5) follows from Part (iv) of Theorem 4.1, with uniform convergence for  $f \in C^\infty(M)$  in view of (4.3). Then, using (4.5) and (3.6) we can calculate

$$\begin{aligned}
(T_\sigma e_j^k)(x) &= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} (\sigma(\ell) \widehat{e_j^k}(\ell))_m e_\ell^m(x) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \left( \sum_{i=1}^{d_\ell} (\sigma(\ell))_{mi} \widehat{e_j^k}(\ell, i) \right) e_\ell^m(x) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \sum_{i=1}^{d_\ell} (\sigma(\ell))_{mi} \delta_{j\ell} \delta_{ki} e_\ell^m(x) \\
&= \sum_{m=1}^{d_j} (\sigma(j))_{mk} e_j^m(x),
\end{aligned}$$

yielding (4.6).  $\square$

Theorem 2.3 characterising invariant operators bounded on  $L^2(M)$  now becomes

**Theorem 4.3.** *An invariant linear operator  $T : C^\infty(M) \rightarrow L^2(M)$  extends to a bounded operator from  $L^2(M)$  to  $L^2(M)$  if and only if its symbol  $\sigma$  satisfies*

$$\sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{op} < \infty,$$

where  $\|\sigma(\ell)\|_{op} = \|\sigma(\ell)\|_{\mathcal{L}(H_\ell)}$  is the matrix multiplication operator norm with respect to  $H_\ell \simeq \ell^2(\mathbb{C}^{d_\ell})$ . Moreover, we have

$$\|T\|_{\mathcal{L}(L^2(M))} = \sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{op}.$$

This can be extended to Sobolev spaces. We will use the multiplication property for Fourier multipliers which is a direct consequence of Proposition 2.4:

**Proposition 4.4.** *If  $S, T : C^\infty(M) \rightarrow L^2(M)$  are invariant operators with respect to  $E$  such that the domain of  $S \circ T$  contains  $C^\infty(M)$ , then  $S \circ T : C^\infty(M) \rightarrow L^2(M)$  is also invariant with respect to  $E$ . Moreover, if  $\sigma_S$  denotes the symbol of  $S$  and  $\sigma_T$  denotes the symbols of  $T$  with respect to the same orthonormal basis then*

$$\sigma_{S \circ T} = \sigma_S \sigma_T,$$

i.e.  $\sigma_{S \circ T}(j) = \sigma_S(j) \sigma_T(j)$  for all  $j \in \mathbb{N}_0$ .

Recalling Sobolev spaces  $H^s(M)$  in (3.8) we have:

**Corollary 4.5.** *Let an invariant linear operator  $T : C^\infty(M) \rightarrow C^\infty(M)$  have symbol  $\sigma_T$  for which there exists  $C > 0$  and  $m \in \mathbb{R}$  such that*

$$\|\sigma_T(\ell)\|_{op} \leq C(1 + \lambda_\ell)^{\frac{m}{\nu}}$$

*holds for all  $\ell \in \mathbb{N}_0$ . Then  $T$  extends to a bounded operator from  $H^s(M)$  to  $H^{s-m}(M)$  for every  $s \in \mathbb{R}$ .*

*Proof.* We note that by (3.8) the condition that  $T : H^s(M) \rightarrow H^{s-m}(M)$  is bounded is equivalent to the condition that the operator

$$S := (I + E)^{\frac{s-m}{\nu}} \circ T \circ (I + E)^{-\frac{s}{\nu}}$$

is bounded on  $L^2(M)$ . By Proposition 4.4 and the fact that the powers of  $E$  are pseudo-differential operators with diagonal symbols, see (4.4), we have

$$\sigma_S(\ell) = (1 + \lambda_\ell)^{-\frac{m}{\nu}} \sigma_T(\ell).$$

But then  $\|\sigma_S(\ell)\|_{op} \leq C$  for all  $\ell$  in view of the assumption on  $\sigma_T$ , so that the statement follows from Theorem 4.3.  $\square$

## 5. SCHATTEN CLASSES OF OPERATORS ON COMPACT MANIFOLDS

In this section we give an application of the constructions in the previous section to determine the membership of operators in Schatten classes and then apply it to a particular family of operators on  $L^2(M)$ .

As a consequence of Theorem 2.5, we can now characterise invariant operators in Schatten classes on compact manifolds. We note that this characterisation does not assume any regularity of the kernel nor of the symbol. Once we observe that the conditions for the membership in the Schatten classes depend only on the basis  $e_j^k$  and not on the operator  $E$ , we immediately obtain:

**Theorem 5.1.** *Let  $0 < r < \infty$ . An invariant operator  $T : L^2(M) \rightarrow L^2(M)$  is in  $S_r(L^2(M))$  if and only if  $\sum_{\ell=0}^{\infty} \|\sigma_T(\ell)\|_{S_r}^r < \infty$ . Moreover*

$$\|T\|_{S_r(L^2(M))}^r = \sum_{\ell=0}^{\infty} \|\sigma_T(\ell)\|_{S_r}^r.$$

*If an invariant operator  $T : L^2(M) \rightarrow L^2(M)$  is in the trace class  $S_1(L^2(M))$ , then*

$$\mathrm{Tr}(T) = \sum_{\ell=0}^{\infty} \mathrm{Tr}(\sigma_T(\ell)).$$

*Remark 5.2.* In Section 6 we will establish a relation between the notion of symbol introduced in Theorem 4.1 and the corresponding symbol in the setting of compact Lie groups (cf. [RT10, RT13]). In particular the characterisation above extends the one obtained in Theorem 3.7 of [DR13].

We now apply Theorem 5.1 to determining which powers of  $E$  belong to which Schatten classes. But first we record a useful relation between the sequences  $\lambda_j$  and  $d_j$  of eigenvalues of  $E$  and their multiplicities.

**Proposition 5.3.** *Let  $M$  be a closed manifold of dimension  $n$ , and let  $E \in \Psi_{+e}^\nu(M)$ , with  $\nu > 0$ . Then there exists a constant  $C > 0$  such that we have*

$$d_j \leq C(1 + \lambda_j)^{\frac{n}{\nu}} \tag{5.1}$$

*for all  $j \geq 1$ . Moreover, we also have*

$$\sum_{j=1}^{\infty} d_j (1 + \lambda_j)^{-q} < \infty \quad \text{if and only if} \quad q > \frac{n}{\nu}. \tag{5.2}$$

*Proof.* Since  $(1 + \lambda_j)^{1/\nu}$  are the eigenvalues of the first-order elliptic positive operator  $(I + E)^{1/\nu}$  with multiplicities  $d_j$ , the Weyl eigenvalue counting formula for the operator  $(I + E)^{1/\nu}$  gives

$$\sum_{j: (1+\lambda_j)^{1/\nu} \leq \lambda} d_j = C_0 \lambda^n + O(\lambda^{n-1})$$

as  $\lambda \rightarrow \infty$ . This implies  $d_j \leq C(1 + \lambda_j)^{n/\nu}$  for sufficiently large  $\lambda_j$ , implying the estimate (5.1).

To prove (5.2), let us denote  $T := (I + E)^{-q/2}$ . Then the eigenvalues of  $T$  are  $(1 + \lambda_j)^{-q/2}$  with multiplicities  $d_j$ . This implies

$$\sum_{j=0}^{\infty} d_j (1 + \lambda_j)^{-q} = \|T\|_{S_2}^2 \asymp \|K\|_{L^2(M \times M)}^2. \quad (5.3)$$

By the functional calculus of pseudo-differential operators, we have  $T \in \Psi^{-\nu q/2}(M)$ , and so its integral kernel  $K(x, y)$  is smooth for  $x \neq y$ , and near the diagonal  $x = y$ , identifying points with their local coordinates, we have

$$|K(x, y)| \leq C_\alpha |x - y|^{-\alpha},$$

for any  $\alpha > n - \nu q/2$ , see e.g. [Dui11] or [RT10, Theorem 2.3.1]. Thus order is sharp with respect to the order of the operator. Therefore,  $K \in L^2(M \times M)$  if and only if there exists  $\alpha$  such that  $n > 2\alpha > 2n - \nu q$ . Together with (5.3) this implies (5.2).  $\square$

**Proposition 5.4.** *Let  $M$  be a closed manifold of dimension  $n$ , and let  $E \in \Psi_{+e}^\nu(M)$  be a positive elliptic pseudo-differential operator of order  $\nu > 0$ . Let  $0 < p < \infty$ . Then we have*

$$(I + E)^{-\frac{\alpha}{\nu}} \in S_p(L^2(M)) \text{ if and only if } \alpha > \frac{n}{p}. \quad (5.4)$$

*Proof.* We note that the operator  $(I + E)^{-\frac{\alpha}{\nu}}$  is positive definite, its singular values are  $(1 + \lambda_j)^{-\frac{\alpha}{\nu}}$  with multiplicities  $d_j$ . Therefore,

$$\|(I + E)^{-\frac{\alpha}{\nu}}\|_{S_p}^p = \sum_{j=0}^{\infty} d_j (1 + \lambda_j)^{-\frac{\alpha p}{\nu}},$$

which is finite if and only if  $\alpha p > n$  by (5.2), implying the statement.  $\square$

## 6. RELATION TO THE SETTING OF COMPACT LIE GROUPS

In the recent work [DR13] the authors studied Schatten classes of operators on compact Lie groups. We now explore how the notion of the symbol from Theorem 4.1 corresponds to the matrix-valued symbols on compact Lie groups, and how the results for Schatten classes correspond to each other when  $M = G$  is a compact Lie group. In this and the following sections we assume that all operators are continuous on  $\mathcal{D}'(G)$  so that the integral kernels of such operators are distributions.

We will give two types of decompositions of  $L^2(G)$  into  $H_j$ 's as in (2.1). First, we choose  $H_j$ 's determined by unitary irreducible representations of  $G$ . However, in this case the condition (3.1) may fail. Consequently, to view this analysis as a special case of the construction on manifolds in Section 4 with condition (3.1), we group representations corresponding to the same eigenvalue of the Laplacian together,

to form a coarser decomposition of  $L^2(G)$  into a direct sum of finite dimensional subspaces. The example of this types of partitions is given in Remark 2.6 in the case of the torus  $\mathbb{T}^n$ .

Now, we recall some basic definitions. Let  $G$  be a compact Lie group of dimension  $n$  equipped with the normalised Haar measure. Let  $\widehat{G}$  denote the set of equivalence classes of continuous irreducible unitary representations of  $G$ . Since  $G$  is compact, the set  $\widehat{G}$  is discrete. For  $[\xi] \in \widehat{G}$ , by choosing a basis in the representation space of  $\xi$ , we can view  $\xi$  as a matrix-valued function  $\xi : G \rightarrow \mathbb{C}^{d_\xi \times d_\xi}$ , where  $d_\xi$  is the dimension of the representation space of  $\xi$ . By the Peter-Weyl theorem the collection

$$\left\{ \sqrt{d_\xi} \xi_{ij} : 1 \leq i, j \leq d_\xi, [\xi] \in \widehat{G} \right\}$$

is the orthonormal basis of  $L^2(G)$ . If  $f \in L^1(G)$  we define its group Fourier transform at  $\xi$  by

$$\mathcal{F}_G f(\xi) \equiv \widehat{f}(\xi) := \int_G f(x) \xi(x)^* dx, \quad (6.1)$$

where  $dx$  is the normalised Haar measure on  $G$ . If  $\xi$  is a matrix representation, we have  $\widehat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$ . We note that this Fourier transform is different from the one we considered on manifolds in (3.3) which produced vector-valued Fourier coefficients instead of the matrix-valued ones obtained in (6.1).

The Fourier inversion formula is a consequence of the Peter-Weyl theorem, so that we have

$$f(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}(\xi(x) \widehat{f}(\xi)). \quad (6.2)$$

For each  $[\xi] \in \widehat{G}$ , the matrix elements of  $\xi$  are the eigenfunctions for the Laplacian  $\mathcal{L}_G$  (or the Casimir element of the universal enveloping algebra), with the same eigenvalues which we denote by  $-\lambda_{[\xi]}^2$ , so that we have

$$-\mathcal{L}_G \xi_{ij}(x) = \lambda_{[\xi]}^2 \xi_{ij}(x) \quad \text{for all } 1 \leq i, j \leq d_\xi. \quad (6.3)$$

For a thorough discussion of Laplacians on compact Lie groups we refer to [Ste70].

The weight for measuring the decay or growth of Fourier coefficients in this setting is  $\langle \xi \rangle := (1 + \lambda_{[\xi]}^2)^{\frac{1}{2}}$ , the eigenvalues of the elliptic first-order pseudo-differential operator  $(I - \mathcal{L}_G)^{\frac{1}{2}}$ . The Parseval identity takes the form

$$\|f\|_{L^2(G)} = \left( \sum_{[\xi] \in \widehat{G}} d_\xi \|\widehat{f}(\xi)\|_{\text{HS}}^2 \right)^{\frac{1}{2}}, \quad \text{where } \|\widehat{f}(\xi)\|_{\text{HS}}^2 = \operatorname{Tr}(\widehat{f}(\xi) \widehat{f}(\xi)^*),$$

which defines the norm on  $\ell^2(\widehat{G})$ .

For a linear continuous operator  $A$  from  $C^\infty(G)$  to  $\mathcal{D}'(G)$  we define its matrix-valued symbol  $\tau_A(x, \xi) \in \mathbb{C}^{d_\xi \times d_\xi}$  by

$$\tau_A(x, \xi) := \xi(x)^* (A\xi)(x) \in \mathbb{C}^{d_\xi \times d_\xi}. \quad (6.4)$$

Then one has ([RT10], [RT13]) the global quantization

$$Af(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}(\xi(x) \tau_A(x, \xi) \widehat{f}(\xi)) \quad (6.5)$$

in the sense of distributions, and the sum is independent of the choice of a representation  $\xi$  from each equivalence class  $[\xi] \in \widehat{G}$ . If  $A$  is a linear continuous operator from  $C^\infty(G)$  to  $C^\infty(G)$ , the series (6.5) is absolutely convergent and can be interpreted in the pointwise sense. We will also write  $A = \text{Op}(\tau_A)$  for the operator  $A$  given by the formula (6.5). We refer to [RT10, RT13] for the consistent development of this quantization and the corresponding symbolic calculus.

In the case of a left-invariant operator  $A$ , its symbol  $\tau_A$  is independent of  $x$ , and formula (6.4) reduces to

$$\tau_A(\xi) = \xi(x)^*(A\xi)(x) = A\xi(e), \quad (6.6)$$

where  $e$  is the unit element of the group.

We can now establish a correspondence between the two frameworks, the one in this paper and the one given in [DR13]. In the setting of compact Lie groups the unitary dual being discrete, we can enumerate the representations as  $\xi_j$  for  $0 \leq j < \infty$ . The indices  $(i, \ell)$  of each matrix  $\xi(x)$  will be enumerated following the lexicographical order  $((i, \ell) \leq (i', \ell')$  if  $i < i'$  or  $(i = i'$  and  $\ell \leq \ell')$ ). In this way, we fix the orthonormal basis  $\{e_j^k\}$  given by

$$\{e_j^k\}_{1 \leq k \leq d_j} = \left\{ \sqrt{d_{\xi_j}}(\xi_j)_{i\ell} \right\}_{1 \leq i, \ell \leq d_{\xi_j}}, \quad (6.7)$$

where  $d_j = d_{\xi_j}^2$  and  $k$  represents an entry of the matrix of the representation following the lexicographical order described above. Then we have the subspaces

$$H_j \equiv H_{[\xi_j]} := \text{span}\{(\xi_j)_{i\ell} : 1 \leq i, \ell \leq d_{\xi_j}\}. \quad (6.8)$$

On a compact Lie group  $G$  we can consider  $E$  to be a bi-invariant Laplacian, see Stein [Ste70] for a discussion of such operators. Then, in view of the Peter-Weyl theorem, the functions  $\{e_j^k\}_{1 \leq k \leq d_j}$  are its eigenfunctions, with norm one in  $L^2(G)$  with respect to the normalised Haar measure, and corresponding to the same eigenvalue  $\lambda_j$ . However, the condition (3.1) does not hold in general since non-equivalent representations in  $\widehat{G}$  may give the same eigenvalues of the Laplacian.

We now observe that there is also a correspondence between the vector-valued Fourier transform introduced in (3.3) and the matrix-valued Fourier transform defined in (6.1). Such correspondence can be established by applying once more the lexicographical order to the matrix-valued Fourier transform (6.1).

In order to study such correspondence, for  $d \in \mathbb{N}$  we will define a bijection from the set of indices of the matrix-symbol  $\{1, \dots, d\}^2$  onto the set of indices  $\{1, \dots, d^2\}$  and calculate its inverse. If  $(j, k) \in \{1, \dots, d\}^2$  we define

$$\Gamma_d(j, k) := (j - 1)d + k.$$

The function  $\Gamma_d$  is surjective, indeed if  $t \in \{1, \dots, d^2\}$ ,  $j$  can be obtained from

$$j = \left\lfloor \frac{t-1}{d} \right\rfloor + 1,$$

where  $\lfloor \cdot \rfloor$  denotes the function defined for  $x \geq 0$  by  $\lfloor x \rfloor = \max\{y \in \mathbb{N}_0 : y \leq x\}$ .

For the term  $k$  we observe that

$$j - 1 = \left\lfloor \frac{t-1}{d} \right\rfloor,$$



hence

$$k = t - \left\lfloor \frac{t-1}{d} \right\rfloor d.$$

Since we are dealing with finite sets with the same number of elements, the injectivity of  $\Gamma$  follows.

We can now establish correspondences between the Fourier transforms on  $G = M$ , for  $M$  viewed as both a compact manifold and a compact Lie group. Taking into account (6.1) and (6.7) we obtain

$$(\mathcal{F}_M f)(i, t) = (f, e_i^t)_{L^2} = \sqrt{d_{\xi_i}} ((\mathcal{F}_G f)(\xi_i))_{(t - \left\lfloor \frac{t-1}{d_{\xi_i}} \right\rfloor d_{\xi_i}, \left\lfloor \frac{t-1}{d_{\xi_i}} \right\rfloor + 1)}, \quad (6.9)$$

for  $i \in \mathbb{N}_0$ ,  $1 \leq t \leq d_i = d_{\xi_i}^2$ . In the another direction we have

$$((\mathcal{F}_G f)(\xi_\ell))_{i,j} = \frac{1}{\sqrt{d_{\xi_\ell}}} (\mathcal{F}_M f)(\ell, \Gamma_{d_{\xi_\ell}}(j, i)), \quad (6.10)$$

for  $1 \leq i, j \leq d_{\xi_\ell}$ .

For the sake of simplicity, we introduce the following notation:

$$\psi(t, d) := \left\lfloor \frac{t-1}{d} \right\rfloor + 1, \quad \phi(t, d) := t - \left\lfloor \frac{t-1}{d} \right\rfloor d,$$

where  $t \in \{1, \dots, d^2\}$ . With this notation formula (6.9) becomes

$$(\mathcal{F}_M f)(\ell, m) = \sqrt{d_{\xi_\ell}} ((\mathcal{F}_G f)(\xi_\ell))_{(\phi(m, d_{\xi_\ell}), \psi(m, d_{\xi_\ell}))}. \quad (6.11)$$

We also have

$$e_j^k = (\sqrt{d_{\xi_j}} \xi_j)_{(\psi(k, d_{\xi_j}), \phi(k, d_{\xi_j}))}.$$

In the calculations below we will use the following basic relations for the Fourier transform on a compact Lie group  $G$ :

$$(\mathcal{F}_G(\eta_{rs})(\eta))_{ij} = \int_G \eta_{rs}(x) \overline{\eta_{ji}(x)} dx = \frac{1}{d_\eta} \delta_{(i,j), (s,r)},$$

which means that  $\mathcal{F}_G(\eta_{rs})(\eta)$  is the matrix of dimension  $d_\eta \times d_\eta$  with the only entry different from zero equal to  $\frac{1}{d_\eta}$  in the position  $(s, r)$ . We will denote this matrix by  $\frac{1}{d_\eta} (\delta_{(i,j), (s,r)})_{ij}$ , and we have also  $\delta_{(i,j), (s,r)} = 1$  if  $i = s$  and  $r = j$ , and  $\delta_{(i,j), (s,r)} = 0$  if  $i \neq s$  or  $r \neq j$ .

Thus, for an invariant operator we obtain

$$(\mathcal{F}_G(T(\xi_{rs})))(\xi) = \tau(\xi) (\mathcal{F}_G(\xi_{rs})(\xi)) = \tau(\xi) \frac{1}{d_\xi} (\delta_{(i,j), (s,r)})_{ij}. \quad (6.12)$$

In other words  $(\mathcal{F}_G(T(\xi_{rs})))(\xi)$  is a matrix of dimension  $d_\xi \times d_\xi$  with all the columns zero except for the  $r$ -column which is equal to the  $s$ -column of  $\frac{1}{d_\xi} \tau(\xi)$ .

We shall denote by  $\sigma$  the symbol corresponding to  $T$  and consider the orthonormal basis  $\{e_j^k\}$  defined in (6.7) in the sense of (4.1) on manifolds. The symbol introduced in (6.4) in the sense of groups will be denoted by  $\tau$ . We now can find formulae relating

the symbols  $\tau$  and  $\sigma$ . We begin by finding a formula for  $\sigma$  in terms of  $\tau$ . By (6.11), (4.1) and (6.12) we obtain

$$\begin{aligned}
\sigma(\ell)_{mi} &= (\mathcal{F}_M(Te_\ell^i))(\ell, m) \\
&= \sqrt{d_{\xi_\ell}}((\mathcal{F}_G(Te_\ell^i))(\xi_\ell))_{(\phi(m, d_{\xi_\ell}), \psi(m, d_{\xi_\ell}))} \\
&= \sqrt{d_{\xi_\ell}}((\mathcal{F}_G(T(\sqrt{d_{\xi_\ell}}\xi_\ell)^{\psi(i, d_{\xi_\ell}), \phi(i, d_{\xi_\ell})}))(\xi_\ell))_{(\phi(m, d_{\xi_\ell}), \psi(m, d_{\xi_\ell}))} \\
&= d_{\xi_\ell}((\mathcal{F}_G(T(\xi_\ell)^{\psi(i, d_{\xi_\ell}), \phi(i, d_{\xi_\ell})}))(\xi_\ell))_{(\phi(m, d_{\xi_\ell}), \psi(m, d_{\xi_\ell}))} \\
&= d_{\xi_\ell} d_{\xi_\ell}^{-1}(\tau(\xi_\ell)(\delta_{((p, q), (\phi(i, d_{\xi_\ell}), \psi(i, d_{\xi_\ell})))pq})_{(\phi(m, d_{\xi_\ell}), \psi(m, d_{\xi_\ell}))} \\
&= \tau(\xi_\ell)_{(\phi(m, d_{\xi_\ell}), \phi(i, d_{\xi_\ell}))} \delta_{\psi(i, d_{\xi_\ell}), \psi(m, d_{\xi_\ell})}.
\end{aligned}$$

Therefore, we obtain

$$\sigma(\ell)_{mi} = \begin{cases} \tau(\xi_\ell)_{(\phi(m, d_{\xi_\ell}), \phi(i, d_{\xi_\ell}))}, & \text{if } \psi(m, d_{\xi_\ell}) = \psi(i, d_{\xi_\ell}), \\ 0, & \text{otherwise.} \end{cases} \quad (6.13)$$

We note that both functions  $\phi$  and  $\psi$  are periodic with respect to the first parameters  $i$  and  $m$ , implying that there is a periodic structure in the ‘big’ manifold-symbol  $\sigma$  composed of some copies of the ‘small’ group-symbol  $\tau$ .

We will now give a graphical description of the relations (6.13) between the two symbols. The entries of  $\tau(\xi_\ell)$  are distributed inside the matrix-symbol  $\sigma$  according to (6.13): setting  $d := d_{\xi_\ell}$  it is

$$\begin{array}{cccccccccccccc}
& & & & i & & & & & & & & \\
& & & d_{\xi_\ell} & d_{\xi_\ell} + 1 & & & & & & & d_{\xi_\ell}^2 & \\
& & & \downarrow & \downarrow & & & & & \downarrow & & \downarrow & \\
\left( \begin{array}{cccccccccccccc}
\tau(\xi_\ell)_{11} & \tau(\xi_\ell)_{12} & \cdots & \tau(\xi_\ell)_{1d} & 0 & 0 & \cdots & 0 & & 0 & 0 & \cdots & 0 \\
\tau(\xi_\ell)_{21} & \tau(\xi_\ell)_{22} & \cdots & \tau(\xi_\ell)_{2d} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\
\tau(\xi_\ell)_{d1} & \tau(\xi_\ell)_{d2} & \cdots & \tau(\xi_\ell)_{dd} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \tau(\xi_\ell)_{11} & \tau(\xi_\ell)_{12} & \cdots & \tau(\xi_\ell)_{1d} & & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \tau(\xi_\ell)_{21} & \tau(\xi_\ell)_{22} & \cdots & \tau(\xi_\ell)_{2d} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & \tau(\xi_\ell)_{d1} & \tau(\xi_\ell)_{d2} & \cdots & \tau(\xi_\ell)_{dd} & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & & \tau(\xi_\ell)_{11} & \tau(\xi_\ell)_{12} & \cdots & \tau(\xi_\ell)_{1d} \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \tau(\xi_\ell)_{21} & \tau(\xi_\ell)_{22} & \cdots & \tau(\xi_\ell)_{2d} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \cdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & \tau(\xi_\ell)_{d1} & \tau(\xi_\ell)_{d2} & \cdots & \tau(\xi_\ell)_{dd}
\end{array} \right)
\end{array}$$

On the other hand, given the symbol  $\sigma$ , an application of equations (6.13) for  $1 \leq m, i \leq d_{\xi_\ell}$  gives

$$\tau(\xi_\ell)_{mi} = \sigma(\ell)_{mi}, \text{ for } 1 \leq m, i \leq d_{\xi_\ell}. \quad (6.14)$$

The proposition below shows that the Schatten quasi-norms  $\|\cdot\|_{S_r}$  of the symbols  $\tau$  and  $\sigma$  are in agreement when  $M = G$  is a compact Lie group. Thus, our results

in Section 5 are an extension of those in [DR13] concerning Schatten classes. In particular, Theorem 5.1 extends Theorem 3.7 of [DR13] as announced in Remark 5.2. We recall that on a compact Lie group  $G$  we take  $E$  to be a bi-invariant Laplacian.

**Proposition 6.1.** *Let  $G$  be a compact Lie group. If a linear operator  $T : C^\infty(G) \rightarrow L^2(G)$ , continuous on  $\mathcal{D}'(G)$ , is left-invariant then it is also invariant relative to the family of  $H_j$ 's as in (6.8) in the sense of Theorem 2.1 (in fact, it is also strongly invariant).*

*Let  $T : C^\infty(G) \rightarrow L^2(G)$  be a left-invariant operator, and let  $\sigma$  be its symbol in the sense of Theorem 2.1 and  $\tau$  its symbol in the sense of groups as in (6.6). Then these symbols are related by formulae (6.13)–(6.14).*

*Consequently, for a bounded left-invariant operator  $T : L^2(G) \rightarrow L^2(G)$ , for every  $0 < r < \infty$  we have*

$$\|\sigma(\ell)\|_{S_r}^r = d_{\xi_\ell} \|\tau(\xi_\ell)\|_{S_r}^r,$$

and, therefore,

$$\sum_{\ell} \|\sigma(\ell)\|_{S_r}^r = \sum_{\ell} d_{\xi_\ell} \|\tau(\xi_\ell)\|_{S_r}^r.$$

*Proof.* The invariance in the sense of groups as in (6.6) of the group-left-invariant operators follows from the relation (6.13) between symbols and from the characterisation in Theorem 2.1.

For the following statements, since for Schatten quasi-norms we have

$$\|B\|_{S_r} = \||B|\|_{S_r},$$

we can assume that  $\sigma, \tau$  are symmetric, and hence they can be also assumed diagonal. On the other hand, using the relation between  $\sigma$  and  $\tau$  in (6.13) and (6.14), and by looking at the diagonal elements of  $\sigma$  in (6.13), we obtain

$$\|\sigma(\ell)\|_{S_r}^r = \sum_{m=1}^{d_{\xi_\ell}^2} |\sigma(\ell)_{mm}|^r = d_{\xi_\ell} \sum_{m=1}^{d_{\xi_\ell}} |\tau(\xi_\ell)_{mm}|^r = d_{\xi_\ell} \|\tau(\xi_\ell)\|_{S_r}^r.$$

Thus  $\|\sigma(\ell)\|_{S_r}^r = d_{\xi_\ell} \|\tau(\xi_\ell)\|_{S_r}^r$  and, therefore,  $\sum_{\ell} \|\sigma(\ell)\|_{S_r}^r = \sum_{\ell} d_{\xi_\ell} \|\tau(\xi_\ell)\|_{S_r}^r$ .  $\square$

We finish this section by describing an adaptation of the above construction to put it in the framework of manifolds as described in Theorem 4.1. In the case of the torus  $\mathbb{T}^n$  this is indicated in Remark 2.6. Recalling the definition of  $H_{[\xi]}$  in (6.8) for each  $[\xi] \in \widehat{G}$ , and the notation  $\lambda_{[\xi]}$  for the eigenvalues as in (6.3), for the sequence  $0 = \lambda_0^2 < \lambda_1^2 < \lambda_2^2 < \dots$  of eigenvalues of  $-\mathcal{L}_G$  counted without multiplicities we set

$$\widetilde{H}_\ell := \bigoplus_{\substack{[\xi] \in \widehat{G} \\ \lambda_{[\xi]} = \lambda_\ell}} H_{[\xi]} = \bigoplus_{\substack{[\xi] \in \widehat{G} \\ \lambda_{[\xi]} = \lambda_\ell}} \text{span}\{\xi_{ik} : 1 \leq i, k \leq d_\xi\}, \quad \ell \in \mathbb{N}_0. \quad (6.15)$$

The family of  $\widetilde{H}_\ell$ 's is the collection of eigenspaces of the elliptic differential operator  $\mathcal{L}_G$  for which the condition (3.1) is satisfied. The symbols  $\sigma$  and  $\tilde{\sigma}$  of an invariant

operator  $T$  with respect to the partitions  $H_j$ 's and  $\widetilde{H}_\ell$ 's, respectively, are related by

$$\widetilde{\sigma}(\ell) = \bigotimes_{\substack{[\xi_j] \in \widehat{G} \\ \lambda_{[\xi_j]} = \lambda_\ell}} \sigma(j), \quad (6.16)$$

with  $\widetilde{\sigma}(\ell) \in \mathbb{C}^{\widetilde{d}_\ell \times \widetilde{d}_\ell}$  and

$$\widetilde{d}_\ell = \sum_{\substack{[\xi_j] \in \widehat{G} \\ \lambda_{[\xi_j]} = \lambda_\ell}} d_j = \sum_{\substack{[\xi_j] \in \widehat{G} \\ \lambda_{[\xi_j]} = \lambda_\ell}} d_{\xi_j}^2.$$

Recalling the relation (6.13) between the symbol  $\sigma$  in the sense of Theorem 2.1 and the group symbol  $\tau$  as in (6.6), given by

$$\sigma(j) \equiv \sigma(\xi_j) = \begin{pmatrix} \tau(\xi_j) & 0 & \cdots & 0 \\ 0 & \tau(\xi_j) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \tau(\xi_j) \end{pmatrix}, \quad (6.17)$$

the formula (6.16) provides the further relation between the symbol  $\widetilde{\sigma}$  in the sense of manifolds (in Theorem 4.1) and the group symbol  $\tau$ . Therefore, if  $\lambda_{[\xi_1]} = \dots = \lambda_{[\xi_m]} = \lambda_\ell$  for non-equivalent representations  $[\xi_1], \dots, [\xi_m] \in \widehat{G}$ , we have

$$\widetilde{\sigma}(\ell) = \begin{pmatrix} \sigma(\xi_1) & 0 & \cdots & 0 \\ 0 & \sigma(\xi_2) & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \sigma(\xi_m) \end{pmatrix}. \quad (6.18)$$

In particular, we obtain

**Corollary 6.2.** *Let  $G$  be a compact Lie group and let  $T : C^\infty(G) \rightarrow L^2(G)$  be a linear operator, continuous on  $\mathcal{D}'(G)$ . If  $T$  is left-invariant then it is also invariant relative to the operator  $\mathcal{L}_G$  (in the sense of Theorem 4.1). The corresponding symbols are related by formulae (6.16)–(6.18).*

## 7. KERNELS OF INVARIANT OPERATORS ON COMPACT MANIFOLDS

In this section we describe invariant operators relative to  $E$  in terms of their kernels. We first observe that if  $T = T_\sigma$  is invariant with symbol  $\sigma$ , expanding Proposition

4.2 we can write

$$\begin{aligned}
T_\sigma f(x) &= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} (\sigma(\ell) \widehat{f}(\ell))_m e_\ell^m(x) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \sum_{k=1}^{d_\ell} \sigma(\ell)_{mk} \widehat{f}(\ell)_k e_\ell^m(x) \\
&= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \sum_{k=1}^{d_\ell} \sigma(\ell)_{mk} e_\ell^m(x) \int_M f(y) \overline{e_\ell^k(y)} dy \\
&= \int_M \left( \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \sum_{k=1}^{d_\ell} \sigma(\ell)_{mk} e_\ell^m(x) \overline{e_\ell^k(y)} \right) f(y) dy.
\end{aligned}$$

Hence, the integral kernel  $K(x, y)$  of  $T_\sigma$  is given by

$$K(x, y) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \sum_{k=1}^{d_\ell} \sigma(\ell)_{mk} e_\ell^m(x) \overline{e_\ell^k(y)}. \quad (7.1)$$

On the other hand we note that

$$\{e_\ell^m \otimes \overline{e_{\ell'}^{m'}}\}_{\ell, \ell' \geq 0}^{1 \leq m \leq d_\ell, 1 \leq m' \leq d_{\ell'}}$$

defines an orthonormal basis of  $L^2(M \times M)$ . If  $T$  is Hilbert-Schmidt on  $L^2(M)$ , not necessarily invariant, then its kernel  $K$  is square integrable and we can write its decomposition in this basis as

$$K(x, y) = \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} \sum_{m=1}^{d_\ell} \sum_{m'=1}^{d_{\ell'}} ((\mathcal{F}_M \otimes \overline{\mathcal{F}_M})K)(\ell, m, \ell', m') e_\ell^m(x) \overline{e_{\ell'}^{m'}(y)}, \quad (7.2)$$

where  $((\mathcal{F}_M \otimes \overline{\mathcal{F}_M})K)(\ell, m, \ell', m')$  denotes the Fourier coefficients of  $K$  with respect to the basis  $\{e_\ell^m \otimes \overline{e_{\ell'}^{m'}}\}$  given by

$$\begin{aligned}
((\mathcal{F}_M \otimes \overline{\mathcal{F}_M})K)(\ell, m, \ell', m') &= (K, e_\ell^m(x) \overline{e_{\ell'}^{m'}(y)})_{L^2(M \times M)} \\
&= \int_{M \times M} K(x, y) \overline{e_\ell^m(x)} e_{\ell'}^{m'}(y) dx dy.
\end{aligned}$$

We observe from (7.1) and (7.2) that  $T$  is invariant relative to  $(E, \{e_\ell^m\}_{\ell \geq 0}^{1 \leq m \leq d_\ell})$  if and only if

$$((\mathcal{F}_M \otimes \overline{\mathcal{F}_M})K)(\ell, m, \ell', m') = \begin{cases} 0, & \ell \neq \ell', \\ \sigma(\ell)_{mm'}, & \ell = \ell'. \end{cases} \quad (7.3)$$

For example, from (7.1) we obtain

$$\begin{aligned}
(K, e_\ell^m(x) \overline{e_{\ell'}^{m'}(y)})_{L^2(M \times M)} &= \int_{M \times M} \left( \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \sum_{i=1}^{d_j} \sigma(j)_{ki} e_j^k(x) \overline{e_j^i(y)} \right) \overline{e_\ell^m(x)} e_{\ell'}^{m'}(y) dx dy \\
&= \sum_{j=0}^{\infty} \sum_{k=1}^{d_j} \sum_{i=1}^{d_j} \sigma(j)_{ki} \int_M e_j^k(x) \overline{e_\ell^m(x)} dx \int_M e_{\ell'}^{m'}(y) \overline{e_j^i(y)} dy \\
&= \begin{cases} 0, & \ell \neq \ell', \\ \sigma(\ell)_{mm'}, & \ell = \ell'. \end{cases}
\end{aligned}$$

We now introduce some notation which will be useful in order to define a suitable setting to study the above Fourier coefficients and the relation between operator's kernel and symbol. Let

$$\Sigma(M \times M) := \left\{ \tilde{\sigma} = (\tilde{\sigma}(\ell, m, \ell', m'))_{0 \leq \ell, \ell' < \infty}^{1 \leq m \leq d_\ell, 1 \leq m' \leq d_{\ell'}} : \tilde{\sigma}(\ell, m, \ell', m') = 0 \text{ if } \ell \neq \ell' \right\},$$

$$\mathcal{K} := \{K \in \mathcal{D}'(M \times M) : K \text{ defines an invariant operator relative to } E\}.$$

We now consider the mapping

$$K \mapsto (\mathcal{F}_M \otimes \overline{\mathcal{F}_M})K$$

from  $\mathcal{K}$  into  $\Sigma(M \times M)$ . We can identify the family of symbols  $\Sigma(M \times M)$  with the matrices  $\bigcup_{\ell} \mathbb{C}^{d_\ell \times d_\ell}$  by letting

$$\tilde{\sigma} \equiv \sigma$$

such that  $\sigma(\ell)_{mm'} = \tilde{\sigma}(\ell, m, \ell, m')$ . In this way we also get the identification

$$\Sigma(M \times M) \simeq \Sigma_M = \Sigma$$

with  $\Sigma$  from (4.2).

If  $1 \leq p < \infty$  we define

$$\ell^p(\Sigma) = \left\{ \sigma \in \Sigma : \sum_{\ell=0}^{\infty} \|\sigma(\ell)\|_{S_p}^p < \infty \right\}.$$

On  $\ell^p(\Sigma)$  we define the norm

$$\|\sigma\|_{\ell^p(\Sigma)} := \left( \sum_{\ell=0}^{\infty} \|\sigma(\ell)\|_{S_p}^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

If  $p = \infty$  we define

$$\ell^\infty(\Sigma) = \left\{ \sigma \in \Sigma : \sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{op} < \infty \right\},$$

and we endow  $\ell^\infty(\Sigma)$  with the norm

$$\|\sigma\|_{\ell^\infty(\Sigma)} := \sup_{\ell \in \mathbb{N}_0} \|\sigma(\ell)\|_{op}.$$

The integral operator with kernel  $K$  will be sometimes denoted by  $T_K$ . We note that in terms of the norms  $\ell^p(\Sigma)$ , for invariant operators Theorem 4.3 can be formulated as

$$T \in \mathcal{L}(L^2(M)) \iff \sigma_T \in \ell^\infty(\Sigma), \quad (7.4)$$

and Theorem 5.1 can be formulated as

$$T \in S_p(L^2(M)) \iff \sigma_T \in \ell^p(\Sigma) \quad (7.5)$$

for  $0 < p < \infty$ .

For the formulation of the following theorem we will use the mixed-norm  $L^p$  spaces  $L_x^{p_1} L_y^{p_2}$  on the manifold  $M$  for  $1 \leq p_1, p_2 \leq \infty$ . A measurable function  $K(x, y)$  is said to belong to  $L_x^{p_1} L_y^{p_2}(M \times M)$  if

$$\| \| K(x, y) \|_{L_y^{p_2}} \|_{L_x^{p_1}} < \infty.$$

On  $L_x^{p_1} L_y^{p_2}(M \times M)$  we consider the norm  $\| \cdot \|_{L_x^{p_1} L_y^{p_2}} := \| \| \cdot \|_{L_y^{p_2}} \|_{L_x^{p_1}}$ . We also define

$$L^{(p_1, p_2)}(M \times M) := L_x^{p_1} L_y^{p_2}(M \times M) \cap L_y^{p_1} L_x^{p_2}(M \times M),$$

endowed with norm

$$\| \cdot \|_{L^{(p_1, p_2)}} := \max\{ \| \cdot \|_{L_x^{p_1} L_y^{p_2}}, \| \cdot \|_{L_y^{p_1} L_x^{p_2}} \}.$$

We note that in general  $L^{(p_1, p_2)} \neq L^{(p_2, p_1)}$ . The basic properties of mixed-norm  $L^p$  spaces for many variables were first studied by Benedek and Panzone in [BP61]. In particular they proved a version of Stein's Interpolation of operators theorem and as a consequence the Riesz-Thorin theorem in that setting. A slight modification allows us to apply the Riesz-Thorin theorem when the operator  $T$  acts from a mixed-norm  $L^p$  space to an  $\ell^p(\Sigma)$ -space.

**Theorem 7.1.** *If  $1 \leq p \leq 2$  and  $K \in \mathcal{K} \cap L^{(p', p)}$ , then  $(\mathcal{F}_M \otimes \overline{\mathcal{F}_M})K \in \ell^{p'}(\Sigma)$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ .*

*Proof.* If  $p = 2$  we have  $p' = 2$ . From

$$K \in \mathcal{K} \cap L_x^2 L_y^2 \cap L_y^2 L_x^2 = \mathcal{K} \cap L_{x,y}^2 \subset L_{x,y}^2$$

we get a Hilbert-Schmidt operator  $T_K$ . On the other hand, by Theorem 5.1 with  $r = 2$ , if  $\sigma$  is the symbol of  $T_K$  then  $\sum_{\ell} \|\sigma(\ell)\|_{S_2}^2 < \infty$ . Hence and by (7.3) we obtain

$$(\mathcal{F}_M \otimes \overline{\mathcal{F}_M})K \in \ell^2(\Sigma).$$

For  $p = 1$  we have  $p' = \infty$ . If

$$K \in \mathcal{K} \cap L_x^\infty L_y^1 \cap L_y^\infty L_x^1,$$

by Schur's Lemma we get  $T_K \in \mathcal{L}(L^r(M))$  for all  $1 \leq r \leq \infty$ . In particular  $T_K \in \mathcal{L}(L^2(M))$  and by Theorem 4.3 the symbol  $\sigma$  of  $T_K$  satisfies

$$\sup_{\ell} \|\sigma(\ell)\|_{op} < \infty.$$

By (7.3) we have

$$\|(\mathcal{F}_M \otimes \overline{\mathcal{F}_M})K\|_{\ell^\infty(\Sigma)} = \sup_{\ell} \|\sigma(\ell)\|_{op}.$$

Hence  $(\mathcal{F}_M \otimes \overline{\mathcal{F}_M})K \in \ell^\infty(\Sigma)$ . We have shown that

$$(\mathcal{F}_M \otimes \overline{\mathcal{F}_M}) : \mathcal{K} \cap L^{(2,2)} \longrightarrow \ell^2(\Sigma)$$

and

$$(\mathcal{F}_M \otimes \overline{\mathcal{F}_M}) : \mathcal{K} \cap L^{(\infty,1)} \longrightarrow \ell^\infty(\Sigma).$$

By the Riesz-Thorin interpolation theorem between  $L^{(r,s)}$  and  $\ell^p(\Sigma)$  spaces (cf. [BP61, Theorem 2]) we obtain

$$(\mathcal{F}_M \otimes \overline{\mathcal{F}_M}) : \mathcal{K} \cap L^{(p_1,p_2)} \longrightarrow \ell^q(\Sigma),$$

with  $\frac{1}{p_1} = \frac{1-\theta}{2} + \frac{\theta}{\infty}$ ,  $\frac{1}{p_2} = \frac{1-\theta}{2} + \frac{\theta}{1}$ ,  $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{\infty}$  for  $0 \leq \theta \leq 1$ . Hence

$$p_1 = \frac{2}{1-\theta}, p_2 = \frac{2}{1+\theta}, q = \frac{2}{1-\theta}.$$

We observe that if  $p = \frac{2}{1+\theta}$  then  $\theta = \frac{2-p}{p}$  and  $\frac{2}{1-\theta} = \frac{p}{p-1} = p'$ . Thus

$$(\mathcal{F}_M \otimes \overline{\mathcal{F}_M}) : \mathcal{K} \cap L^{(p',p)} \longrightarrow \ell^{p'}(\Sigma),$$

completing the proof.  $\square$

The following corollary is an immediate consequence of Theorems 7.1 and 5.1, it furnishes a sufficient kernel condition for Schatten classes with index  $p' \geq 2$ .

**Corollary 7.2.** *If  $1 \leq p \leq 2$  and  $K \in \mathcal{K} \cap L^{(p',p)}(M \times M)$ , then  $T_K \in S_{p'}(L^2(M))$ .*

We recall that sufficient conditions of the type above in terms of kernels are not allowed for  $0 < p' < 2$  as a consequence of a Carleman's example. Corollary 7.2 is known for general integral operators (cf. [Rus74, Theorem 3]). Here we have deduced a particular version for invariant operators with a simple proof by applying the notion of symbol.

We now describe another representation of the kernel as the ‘generalised’ Fourier transform of the symbol. From formula (7.1) we have

$$\begin{aligned} K(x, y) &= \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \sum_{k=1}^{d_\ell} \sigma(\ell)_{mk} e_\ell^m(x) \overline{e_\ell^k(y)} \\ &= \sum_{\ell=0}^{\infty} \text{Tr}(e_\ell(x)^\top \sigma(\ell) \overline{e_\ell(y)}) \\ &= \sum_{\ell=0}^{\infty} \text{Tr}(\sigma(\ell) \overline{e_\ell(y)} e_\ell(x)^\top) \\ &= \sum_{\ell=0}^{\infty} \text{Tr}(\sigma(\ell) Q_\ell(x, y)), \end{aligned}$$

where

$$Q_\ell(x, y) = \overline{e_\ell(y)} e_\ell(x)^\top \in \mathbb{C}^{d_\ell \times d_\ell}.$$

We notice that the matrix-valued function

$$(Q_\ell(x, y))_{mk} = \overline{e_\ell^m(y)} e_\ell^k(x)$$

is of rank one for every  $\ell$ . Indeed,  $(Q_\ell(x, y))_{mk}$  is nothing else but the tensor product of the vectors  $e_\ell(x), \overline{e_\ell(y)} \in \mathbb{C}^{d_\ell}$ . Since on a normed space  $F$  we have  $\|u \otimes v\|_{op} = \|u\|_F \|v\|_F$ , we get

$$\|Q_\ell(x, y)\|_{op} = \|e_\ell(x)\|_{\ell^2(\mathbb{C}^{d_\ell})} \|e_\ell(y)\|_{\ell^2(\mathbb{C}^{d_\ell})}.$$



From (7.2) we have

$$\sigma(\ell) = \int_{M \times M} K(x, y) Q_\ell(x, y)^* dx dy.$$

Hence

$$\begin{aligned} \|\sigma(\ell)\|_{op} &\leq \|K\|_{L^1(M \times M)} \sup_{x, y} \|Q_\ell(x, y)^*\|_{op} \\ &= \|K\|_{L^1(M \times M)} \sup_{x, y} \|e_\ell(x)\|_{\ell^2(\mathbb{C}^{d_\ell})} \|e_\ell(y)\|_{\ell^2(\mathbb{C}^{d_\ell})}. \end{aligned}$$

*Remark 7.3.* We point out that the mere condition  $K \in L^1(M \times M)$  does not guarantee the  $L^2$  boundedness of the corresponding integral operator  $T$ . Indeed, consider  $M = \mathbb{T}^1$ ,  $g \in L^1(\mathbb{T}^1) \setminus L^2(\mathbb{T}^1)$ ,  $h \equiv 1 \in L^1(\mathbb{T}^1)$ , and the kernel

$$K(\theta, \phi) := g(\theta)h(\phi) \in L^1(\mathbb{T}^1 \times \mathbb{T}^1).$$

It is easy to see that the kernel  $K(\theta, \phi)$  does not define an operator from  $L^2(\mathbb{T}^1)$  into  $L^2(\mathbb{T}^1)$ . For example, with  $f = 1 \in L^2(\mathbb{T}^1)$  we have

$$(T1)(\theta) = g(\theta) \int_{\mathbb{T}^1} h(\phi) d\phi = g(\theta) \notin L^2(\mathbb{T}^1).$$

## 8. APPLICATIONS TO THE NUCLEARITY OF OPERATORS IN $L^p(M)$

We now turn to the study of nuclearity in  $L^p$ -spaces on closed manifolds. Sufficient conditions for  $r$ -nuclearity on  $L^p$  on compact Lie groups have been established in [DR14b]. The study of nuclearity on  $L^p$  in this section relies on the analysis of suitable kernel decompositions and the relation between kernels and symbols described in Section 7.

Let  $E$  and  $F$  be two Banach spaces and  $0 < r \leq 1$ , a linear operator  $T$  from  $E$  into  $F$  is called  $r$ -nuclear if there exist sequences  $(x'_n)$  in  $E'$  and  $(y_n)$  in  $F$  so that

$$Tx = \sum_n \langle x, x'_n \rangle y_n \quad \text{and} \quad \sum_n \|x'_n\|_{E'}^r \|y_n\|_F^r < \infty. \quad (8.1)$$

When  $r = 1$  they are known as *nuclear operators*, in that case this definition agrees with the concept of trace class operator in the setting of Hilbert spaces ( $E = F = H$ ). More generally, Oloff proved in [Olo72] that the class of  $r$ -nuclear operators coincides with the Schatten class  $S_r(H)$  when  $E = F = H$  and  $0 < r \leq 1$ .

The concept of  $r$ -nuclearity was introduced by Grothendieck [Gro55], and it has application to questions of the distribution of eigenvalues of operators in Banach spaces via e.g. the Grothendieck-Lidskii formula. We refer to [DR14b] for several conclusions in the setting of compact Lie groups concerning summability and distribution of eigenvalues of operators on  $L^p$ -spaces once we have information on their  $r$ -nuclearity. Since these arguments are then purely functional analytic, they apply equally well in the present setting of closed manifolds; we omit the repetition but refer the reader to [DR14b] for several relevant applications.

The  $r$ -nuclear operators on Lebesgue spaces are characterised by the following theorem (cf. [Del10]). In the statement below we consider  $(\Omega_1, \mathcal{M}_1, \mu_1)$  and  $(\Omega_2, \mathcal{M}_2, \mu_2)$  to be two  $\sigma$ -finite measure spaces.

**Theorem 8.1.** *Let  $1 \leq p_1, p_2 < \infty$ ,  $0 < r \leq 1$  and let  $q_1$  be such that  $\frac{1}{p_1} + \frac{1}{q_1} = 1$ . An operator  $T : L^{p_1}(\mu_1) \rightarrow L^{p_2}(\mu_2)$  is  $r$ -nuclear if and only if there exist sequences  $(g_n)_n$  in  $L^{p_2}(\mu_2)$ , and  $(h_n)_n$  in  $L^{q_1}(\mu_1)$  such that  $\sum_{n=1}^{\infty} \|g_n\|_{L^{p_2}}^r \|h_n\|_{L^{q_1}}^r < \infty$ , and such that for all  $f \in L^{p_1}(\mu_1)$  we have*

$$Tf(x) = \int \left( \sum_{n=1}^{\infty} g_n(x) h_n(y) \right) f(y) d\mu_1(y), \quad \text{for a.e. } x.$$

In order to study nuclearity on  $L^p(M)$  spaces for a given compact manifold  $M$  of dimension  $n$ , we introduce a function  $\Lambda(j, k; n, p)$  which controls the  $L^p$ -norms of the family of eigenfunctions  $\{e_j^k\}$  of the operator  $E$ , i.e. we will suppose that  $\Lambda(j, k; n, p)$  is such that we have the estimates

$$\|e_j^k\|_{L^p(M)} \leq \Lambda(j, k; n, p). \quad (8.2)$$

In particular, if  $\Lambda$  is such a function we observe that

$$\|e_j^k\|_{L^p(M)} \leq \text{vol}(M)^{\frac{1}{p}} \Lambda(j, k; n, \infty).$$

When  $M = G$  is a compact Lie group efficient  $\|e_j^k\|_{L^p(G)}$  bounds can be obtained (cf. [DR14b]). The estimation of  $L^p$  norms for eigenfunctions of differential elliptic operators on general closed manifolds has been largely studied, see for instance [SZ02]. Some examples will be given at the end of this section. An example can be also obtained from the following simple lemma:

**Lemma 8.2.** *Let  $f$  be such that  $\|f\|_{L^2(M)} = 1$ , then*

- (i)  $\|f\|_{L^p(M)} \leq (\text{vol}(M))^{\frac{2-p}{2p}}$  if  $1 \leq p \leq 2$ .
- (ii)  $\|f\|_{L^p(M)} \leq \|f\|_{L^\infty(M)}^{\frac{p-2}{p}}$  if  $2 \leq p < \infty$ .

*Proof.* (i) By Hölder inequality we have

$$\int_M |f(x)|^p dx \leq \left( \int_M |f(x)|^{p \frac{2}{p-2}} dx \right)^{\frac{p-2}{2}} \left( \int_M |1|^{p \frac{2}{2-p}} dx \right)^{\frac{2-p}{2}} = (\text{vol}(M))^{\frac{2-p}{2}}.$$

(ii) We also have

$$\int_M |f(x)|^p dx = \int_M |f(x)|^{p-2} |f(x)|^2 dx \leq \|f\|_{L^\infty(M)}^{p-2},$$

completing the proof. □

We now formulate a sufficient condition for the  $r$ -nuclearity on  $L^p(M)$  spaces as an application of the notion of the matrix-symbol on closed manifolds. Inspired by Lemma 8.2, we will use the following function  $\tilde{p}$  for  $1 \leq p \leq \infty$ :

$$\tilde{p} := \begin{cases} 0, & \text{if } 1 \leq p \leq 2, \\ \frac{p-2}{p}, & \text{if } 2 < p < \infty, \\ 1, & \text{if } p = \infty. \end{cases} \quad (8.3)$$

For  $p_1, p_2$  we denote their dual indices by  $q_1 := p'_1$ ,  $q_2 := p'_2$ .

**Theorem 8.3.** *Let  $1 \leq p_1, p_2 < \infty$  and  $0 < r \leq 1$ . Let  $T : L^{p_1}(M) \rightarrow L^{p_2}(M)$  be a strongly invariant linear continuous operator. Assume that its matrix-valued symbol  $\sigma(\ell)$  satisfies*

$$\sum_{\ell=0}^{\infty} \sum_{m,k=1}^{d_\ell} |\sigma(\ell)_{mk}|^r \Lambda(\ell, m; n, \infty)^{\tilde{p}_2 r} \Lambda(\ell, k; n, \infty)^{\tilde{q}_1 r} < \infty.$$

*Then the operator  $T : L^{p_1}(M) \rightarrow L^{p_2}(M)$  is  $r$ -nuclear.*

*Proof.* By (7.1) the kernel of  $T$  is given by

$$K(x, y) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} \sum_{k=1}^{d_\ell} \sigma(\ell)_{mk} e_\ell^m(x) \overline{e_\ell^k(y)}.$$

We set

$$g_{\ell,m,k}(x) := \sigma(\ell)_{mk} e_\ell^m(x), \quad h_{\ell,k}(y) := \overline{e_\ell^k(y)}.$$

Now, by Lemma 8.2 we have

$$\|e_\ell^m\|_{L^p} \leq C_p \Lambda(\ell, m; n, \infty)^{\tilde{p}},$$

where  $C_p = \max\{(vol(M))^{\frac{2-p}{2p}}, 1\}$ . We now observe that

$$\begin{aligned} \sum_{\ell,m,k} \|g_{\ell,m,k}\|_{L^{p_2}}^r \|h_{\ell,k}\|_{L^{q_1}}^r &= \sum_{\ell=0}^{\infty} \sum_{m,k=1}^{d_\ell} \|\sigma(\ell)_{mk} e_\ell^m\|_{L^{p_2}}^r \|\overline{e_\ell^k}\|_{L^{q_1}}^r \\ &= \sum_{\ell=0}^{\infty} \sum_{m,k=1}^{d_\ell} |\sigma(\ell)_{mk}|^r \|e_\ell^m\|_{L^{p_2}}^r \|e_\ell^k\|_{L^{q_1}}^r \\ &\leq (C_{p_2} C_{q_1})^r \sum_{\ell=0}^{\infty} \sum_{m,k=1}^{d_\ell} |\sigma(\ell)_{mk}|^r \Lambda(\ell, m; n, \infty)^{\tilde{p}_2 r} \Lambda(\ell, k; n, \infty)^{\tilde{q}_1 r}, \end{aligned}$$

finishing the proof in view of Theorem 8.1.  $\square$

In particular for formally self-adjoint invariant operators we can diagonalise each matrix  $\sigma(\ell)$ , so that we have

**Corollary 8.4.** *Let  $1 \leq p_1, p_2 < \infty$  and  $0 < r \leq 1$ . Let  $T : L^{p_1}(M) \rightarrow L^{p_2}(M)$  be a strongly invariant formally self-adjoint continuous operator. Assume that its matrix-valued symbol  $\sigma(\ell)$  satisfies*

$$\sum_{\ell=0}^{\infty} \sum_{m=1}^{d_\ell} |\sigma(\ell)_{mm}|^r \Lambda(\ell, m; n, \infty)^{(\tilde{p}_2 + \tilde{q}_1)r} < \infty.$$

*Then the operator  $T : L^{p_1}(M) \rightarrow L^{p_2}(M)$  is  $r$ -nuclear.*

In some cases it is possible to simplify the sufficient condition above when the control function  $\Lambda(\ell, m; n, \infty)$  is independent of  $m$ . For instance a classical result (local Weyl law) due to Hörmander ([Hör68, Theorem 5.1], [Hör85b, Chapter XXIX]) implies the following estimate:

**Lemma 8.5.** *Let  $M$  be a closed manifold of dimension  $n$ . Let  $E \in \Psi_{+e}^\nu(M)$ , then*

$$\|e_\ell^m\|_{L^\infty} \leq C\lambda_\ell^{\frac{n-1}{2\nu}}. \quad (8.4)$$

*Proof.* In order to explain this estimate we first consider the family of eigenvalues  $\{\lambda_\ell\}$  of  $E$  ordered in the increasing order

$$0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_\ell \leq \dots$$

and counted with multiplicity. For the projection  $P_\ell(f)$  onto  $H_\ell$ , consider  $E_\lambda f := \sum_{\lambda_\ell \leq \lambda} P_\ell(f)$  the associated partial sum operators. Its kernel is given by

$$E_\lambda(x, y) = \sum_{\lambda_\ell \leq \lambda} \sum_{m=1}^{d_\ell} e_\ell^m(x) \overline{e_\ell^m(y)}.$$

If  $p(x, \xi)$  is the principal symbol of  $E$ , by Theorem 5.1 of [Hör68] we have

$$E_\lambda(x, x) = \sum_{\lambda_\ell \leq \lambda} \sum_{m=1}^{d_\ell} |e_\ell^m(x)|^2 = (2\pi)^{-n} \int_{p(x, \xi) \leq \lambda} d\xi + R(x, \lambda) \quad (8.5)$$

with

$$|R(x, \lambda)| \leq C\lambda^{\frac{n-1}{2\nu}}, \quad x \in M.$$

Since  $E_\mu(x, x)$  is increasing right-continuous with respect to  $\mu$ , the fact that the spectrum of  $E$  is discrete, by the continuity of  $\int_{p(x, \xi) \leq \mu} d\xi$  with respect to  $\mu$  and by taking left-hand limit in (8.5) we obtain

$$\lim_{\mu \rightarrow \lambda^-} E_\mu(x, x) = \sum_{\lambda_\ell < \lambda} \sum_{m=1}^{d_\ell} |e_\ell^m(x)|^2 = (2\pi)^{-n} \int_{p(x, \xi) \leq \lambda} d\xi + R(x, \lambda^-).$$

Hence

$$E_{\lambda_\ell}(x, x) - E_{\lambda_\ell^-}(x, x) = \sum_{m=1}^{d_\ell} |e_\ell^m(x)|^2 = R(x, \lambda_\ell) - R(x, \lambda_\ell^-).$$

In particular, we have

$$|e_\ell^m(x)| \leq 2(\sqrt{R(x, \lambda_\ell)} + \sqrt{R(x, \lambda_\ell^-)}) \leq 2C\lambda_\ell^{\frac{n-1}{2\nu}},$$

which proves Lemma 8.5.  $\square$

Thus  $\Lambda(\ell; n, \infty) = C\lambda_\ell^{\frac{n-1}{2\nu}}$  furnishes an example of  $\Lambda$  independent of  $m$ . For controls of type  $\Lambda(\ell; n, \infty)$  we have a basis-independent condition:

**Corollary 8.6.** *Let  $1 \leq p_1, p_2 < \infty$  and  $0 < r \leq 1$ . Let  $T : L^{p_1}(M) \rightarrow L^{p_2}(M)$  be a strongly invariant formally self-adjoint continuous operator. Assume that its matrix-valued symbol  $\sigma(\ell)$  satisfies*

$$\sum_{\ell=0}^{\infty} \|\sigma(\ell)\|_{S_r}^r \Lambda(\ell; n, \infty)^{(\tilde{p}_2 + \tilde{q}_1)r} < \infty.$$

Then the operator  $T : L^{p_1}(M) \rightarrow L^{p_2}(M)$  is  $r$ -nuclear. In particular, if its matrix-valued symbol  $\sigma(\ell)$  satisfies

$$\sum_{\ell=0}^{\infty} \|\sigma(\ell)\|_{S_r}^r \lambda_{\ell}^{\frac{(n-1)}{2\nu}(\tilde{p}_2 + \tilde{q}_1)r} < \infty, \quad (8.6)$$

then the operator  $T : L^{p_1}(M) \rightarrow L^{p_2}(M)$  is  $r$ -nuclear.

*Proof.* Since  $T$  is  $E$ -invariant and formally self-adjoint, each matrix  $\sigma(\ell)$  can be assumed diagonal, and the result follows from Corollary 8.4 since

$$\sum_{m=1}^{d_{\ell}} |\sigma(\ell)_{mm}|^r = \text{Tr}(|\sigma(\ell)|^r) = \|\sigma(\ell)\|_{S_r}^r,$$

completing the proof. The  $r$ -nuclearity under condition (8.6) follows by using Lemma 8.5 and taking  $\Lambda(\ell; n, \infty) = C\lambda_{\ell}^{\frac{n-1}{2\nu}}$ .  $\square$

*Remark 8.7.* If  $M$  is a compact Lie group Corollary 8.6 absorbs Theorem 3.4 in [DR14b] by taking  $E$  to be the Laplacian and the family of eigenfunctions  $\{e_{\ell}^k\}$  as in (6.7). Indeed, since  $|d_{\xi_{\ell}}^{\frac{1}{2}}(\xi_{\ell})_{ij}(x)| \leq d_{\xi_{\ell}}^{\frac{1}{2}}$  one can choose  $\Lambda(\ell; \infty) = d_{\xi_{\ell}}^{\frac{1}{2}}$  and taking into account that, by Lemma 6.1, we have

$$\|\sigma(\ell)\|_{S_r}^r = d_{\xi_{\ell}} \|\tau(\xi_{\ell})\|_{S_r}^r,$$

we obtain

$$\sum_{\ell} \|\sigma(\ell)\|_{S_r}^r \Lambda(\ell; \infty)^{(\tilde{p}_2 + \tilde{q}_1)r} = \sum_{\ell} d_{\xi_{\ell}}^{1 + \frac{1}{2}(\tilde{p}_2 + \tilde{q}_1)r} \|\tau(\xi_{\ell})\|_{S_r}^r,$$

with a right-hand side equivalent to the term giving the sufficient condition in Theorem 3.4 of [DR14b]. Indeed,

$$\begin{aligned} \frac{1}{2}(\tilde{p}_2 + \tilde{q}_1) &= \frac{1}{2} \left( 1 - \frac{2}{\max\{2, p_2\}} + 1 - \frac{2}{\max\{2, q_1\}} \right) \\ &= 1 - \frac{1}{\max\{2, q_1\}} - \frac{1}{\max\{2, p_2\}} \\ &= \frac{1}{\min\{2, p_1\}} - \frac{1}{\max\{2, p_2\}}, \end{aligned}$$

which was the order obtained in [DR14b, Theorem 3.4] on compact Lie groups.

In order to give another example we recall Proposition 5.3 with useful relations between the eigenvalues  $\lambda_j$  and their multiplicities  $d_j$ . As a consequence of Corollary 8.6 and Proposition 5.3, for the negative powers of the operator  $E$  itself we obtain:

**Corollary 8.8.** *Let  $1 \leq p_1, p_2 < \infty$  and  $0 < r \leq 1$ . Let  $E \in \Psi_{+e}^{\nu}(M)$ . If*

$$\alpha > \frac{n}{r} + (\tilde{p}_2 + \tilde{q}_1) \frac{n-1}{2}$$

*then the operator  $(I + E)^{-\frac{\alpha}{\nu}} : L^{p_1}(M) \rightarrow L^{p_2}(M)$  is  $r$ -nuclear.*

Note that if  $p_1 = p_2 = 2$ , we have  $\tilde{p}_2 = \tilde{q}_1 = 0$ , and since Schatten class  $S_r$  and  $r$ -nuclear class coincide on  $L^2(M)$ , Proposition 5.4 shows that the statement of Corollary 8.8 is sharp in this case of indices. However, it does depend on the bounds for eigenvalues which can be improved in the presence of additional structures as discussed in Remark 8.9.

*Proof of Corollary 8.8.* If we denote by  $\lambda_\ell$  the eigenvalues of  $E$ , for  $\alpha > 0$  we observe that  $\sigma_{(I+E)^{-\frac{\alpha}{\nu}}}(\ell) = (1 + \lambda_\ell)^{-\frac{\alpha}{\nu}} I_{d_\ell}$ . Then

$$\|\sigma_{(I+E)^{-\frac{\alpha}{\nu}}}(\ell)\|_{S_r}^r = (1 + \lambda_\ell)^{-\frac{\alpha r}{\nu}} d_\ell.$$

Now by applying Corollary 8.6 we obtain

$$\begin{aligned} \sum_{\ell} \|\sigma(\ell)\|_{S_r}^r \lambda_\ell^{\frac{(n-1)}{2\nu}(\tilde{p}_2 + \tilde{q}_1)r} &\leq C \sum_{\ell} d_\ell (1 + \lambda_\ell)^{-\frac{\alpha r}{\nu}} (1 + \lambda_\ell)^{(\tilde{p}_2 + \tilde{q}_1)\frac{(n-1)r}{2\nu}} \\ &= C \sum_{\ell} d_\ell (1 + \lambda_\ell)^{(-\alpha + (\tilde{p}_2 + \tilde{q}_1)\frac{(n-1)}{2})\frac{r}{\nu}} < \infty, \end{aligned}$$

if  $q = (\alpha - (\tilde{p}_2 + \tilde{q}_1)\frac{(n-1)}{2})\frac{r}{\nu} > \frac{n}{\nu}$  by Proposition 5.3. But this is equivalent to the condition  $\alpha > \frac{n}{r} + (\tilde{p}_2 + \tilde{q}_1)\frac{n-1}{2}$ .  $\square$

*Remark 8.9.* As we pointed out in Remark 8.7, on compact Lie groups we can always choose  $E$  to be a Laplacian with an orthonormal basis given by rescaled matrix elements of representations, for which we can take  $\Lambda(\ell; \infty) = d_{\xi_\ell}^{\frac{1}{2}} = d_\ell^{\frac{1}{4}}$ . At the same time, if  $E$  is an operator of second order (so that  $\nu = 2$ ) the best we can hope for on closed manifolds in general is  $\Lambda(\ell; n, \infty) = C\lambda_\ell^{\frac{n-1}{4}}$  given by Lemma 8.5. In view of (5.1), we always have  $d_\ell^{\frac{1}{4}} \lesssim \lambda_\ell^{\frac{n}{8}}$ , so that this choice on compact Lie groups is better than the general bound  $\Lambda(\ell; n, \infty) = C\lambda_\ell^{\frac{n-1}{4}}$  above. Partly, this is explained by the presence of the additional (group) structure in this case. The other point is that there is a difference in finding  $L^\infty$ -estimates for elements of *any* orthonormal basis as opposed to estimates for a favourable one that may exist due to additional assumptions or structures. However, the latter one seems to be the question much less studied in the literature, see [SZ02] or [TZ02] for some partial discussions.

We now give an example of the above remark in the case of the sphere  $\mathbb{S}^3 \simeq \text{SU}(2)$ . We consider the Laplacian (the Casimir element)  $E = -\mathcal{L}_{\mathbb{S}^3}$ . We will apply the condition given by Theorem 8.6 along with the control  $\Lambda(\ell, \infty) = d_\ell^{\frac{1}{4}}$ . For the symbol of  $(I + E)^{-\frac{\alpha}{2}}$ , since the eigenvalues of  $I + E$  are of the form  $(1 + \ell)\ell$  we obtain

$$\|\sigma_{(I+E)^{-\frac{\alpha}{2}}}(\ell)\|_{S_r}^r = ((1 + \ell)\ell)^{-\frac{\alpha r}{2}} d_\ell \approx ((1 + \ell)\ell)^{-\frac{\alpha r}{2}} \ell^2 \approx (1 + \ell^2)^{1 - \frac{\alpha r}{2}}.$$

Therefore, using  $d_\ell \approx \ell^2$ ,

$$\begin{aligned} \sum_{\ell} \|\sigma_{(I+E)^{-\frac{\alpha}{2}}}(\ell)\|_{S_r}^r \Lambda(\ell, \infty)^{(\tilde{p}_2 + \tilde{q}_1)r} &\leq \sum_{\ell} (1 + \ell^2)^{1 - \frac{\alpha r}{2}} \ell^{\frac{1}{2}(\tilde{p}_2 + \tilde{q}_1)r} \\ &\approx \sum_{\ell} (1 + \ell)^{2 - \alpha r + \frac{1}{2}(\tilde{p}_2 + \tilde{q}_1)r}. \end{aligned}$$

The series on the right-hand side converges if and only if  $2 - \alpha r + \frac{1}{2}(\tilde{p}_2 + \tilde{q}_1)r < -1$ . Thus, the condition

$$\alpha > \frac{3}{r} + \frac{1}{2}(\tilde{p}_2 + \tilde{q}_1)$$

ensures the membership of  $(I + E)^{-\frac{\alpha}{2}}$  in the Schatten class of order  $r$ . Summarising, we have proved the following:

**Corollary 8.10.** *If  $\alpha > \frac{3}{r} + \frac{1}{2}(\tilde{p}_2 + \tilde{q}_1)$ ,  $0 < r \leq 1$ , the operator  $(I - \mathcal{L}_{\mathbb{S}^3})^{-\frac{\alpha}{2}}$  is  $r$ -nuclear from  $L^{p_1}(\mathbb{S}^3)$  into  $L^{p_2}(\mathbb{S}^3)$ .*

Corollary 8.10 gives a direct proof of Corollary 3.19 in [DR14b] which was proved there in the group setting.

*Remark 8.11.* It is clear that the sharpness of the sufficient conditions obtained in this section depends on how sharp is the  $\Lambda$  function we can choose. For instance the best situation for  $\Lambda(\ell, \infty)$  is when it can be chosen constant, i.e. when the eigenfunctions are uniformly bounded. This is the case of the torus  $\mathbb{T}^n$  and unfortunately may be essentially the only one, see [TZ02].

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